A lattice framework for pricing display advertisement options with the stochastic volatility underlying model

CrossMark

Bowei Chen *, Jun Wang

University College London, Gower Street, London, WC1E 6BT, United Kingdom

A B S T R A C T

Advertisement (abbreviated ad) options are a recent development in online advertising. Simply, an ad option is a first look contract in which a publisher or search engine grants an advertiser a right but not obligation to enter into transactions to purchase impressions or clicks from a specific ad slot at a pre-specified price on a specific delivery date. Such a structure provides advertisers with more flexibility of their guaranteed deliveries. The valuation of ad options is an important topic and previous studies on ad options pricing have been mostly restricted to the situations where the underlying prices follow a geometric Brownian motion (GBM). This assumption is reasonable for sponsored search; however, some studies have also indicated that it is not valid for display advertising. In this paper, we address this issue by employing a stochastic volatility (SV) model and discuss a lattice framework to approximate the proposed SV model in option pricing. Our developments are validated by experiments with real advertising data: (i) we find that the SV model has a better fitness over the GBM model; (ii) we validate the proposed lattice model via two sequential Monte Carlo simulation methods; (iii) we demonstrate that advertisers are able to flexibly manage their guaranteed deliveries by using the proposed options, and publishers can have an increased revenue when some of their inventories are sold via ad options.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

Options have been widely used in many fields: financial options are an important derivative when speculating profits as well as hedging risk (Wilmott 2006); real options are an effective decision-making tool for business projects valuation and corporate risk management (Boer 2002). Recently, options have been introduced into the field of online advertising to solve the so called non-guaranteed delivery problem as well as to provide advertisers with greater flexibility in purchasing premium ad inventories. Moon and Kwon (2010) proposed an ad option for advertisers to make a flexible choice of payment at either cost-per-mille (CPM) or cost-per-click (CPC). They are two popular online advertising payment schemes: the former allows an advertiser to pay when his ad is displayed 1000 times to online users while with the latter an advertiser pays only when his ad is clicked by an online user. The proposal of Moon and Kwon (2010) was similar to an *option paying the worst and cash* (Zhang 1998) because the option payoff depends on the minimum difference between CPM and CPC. Wang and Chen (2012) proposed a simple European ad option

between buying and non-buying the impressions that will be created in the future, and discussed the option pricing based on the *one-step binomial lattice method* (Sharpe 1978). Their ad option was priced from the perspective of a risk-averse publisher who wants to hedge the expected revenue in the next step. Chen et al. (2015) investigated a special option for sponsored search whereby an advertiser can target a set of keywords for a certain number of total clicks in the future. Each candidate keyword can be specified with a fixed payment price and the option buyer can exercise the option multiple times at any time prior to or on the contract expiration date. Their design was a generalization of the *dual-strike call option* (Zhang 1998) and the *multi-exercise option* (Marshall 2012).

In this paper, we discuss an ad option that gives an advertiser a right but not obligation to purchase the future impressions or clicks from a specific ad slot (or user tag or keyword) at a pre-specified price. The pre-specified price is also called the *strike price*, which can be same or different to the payment scheme of its underlying ad format. For example, the underlying price (i.e., the winning payment price) of a display impression from real-time bidding (RTB) is usually measured by CPM while the proposed ad option can be specified with a strike price in terms of CPC for this impression. The publisher or search engine who grants this right in exchange for a certain amount of upfront fee, is called the *option price*. Obviously, ad options are more flexible than

* Corresponding author.
E-mail addresses: bowei.chen@cs.ucl.ac.uk (B. Chen), jun.wang@cs.ucl.ac.uk (J. Wang).

guaranteed contracts (Bharadwaj et al. 2010) as on the delivery date. If the advertiser thinks that the spot market is more beneficial, he can join RTB as a bidder and his cost of not using an ad option is only the option price. A contract with a such structure is also called a *first look at inventory* (shortly first look) contract or tactic (Interactive Advertising Bureau of Canada 2015). It means that an advertiser is given the opportunity to buy inventories which a publisher offers to him, and if he has no use for it, it can be sold onto another ad network. The ad options proposed by our this study and Wang and Chen (2012), and Chen et al. (2015) are first look contracts while the ad option studied by Moon and Kwon (2010) is not a first look contract.

When pricing an ad option, the previous research is mostly restricted in their usage to those situations where the underlying price follows a *geometric Brownian motion (GBM)* Samuelson 1965. According to Yuan et al. (2013, 2014) and Chen et al. (2014), there is only a very small number of ad inventories whose CPM or CPC satisfies this assumption. Therefore, the previous studies fail to provide an effective unified framework that covers general situations. In this paper, we address the issue and provide a more general pricing framework. We use a *stochastic volatility (SV) model* to describe the underlying price movement for cases where the GBM assumption is not valid. Based on the SV model, a censored binomial lattice is then constructed for option pricing. We also examine several previous binomial and trinomial lattice methods to price an ad option whose underlying inventory prices follow a GBM model, and deduce the close-form solutions to examine their convergence performance. Our developments are validated by experiments using real advertising data. We examine the fitness of the underlying model, valid the proposed option pricing method, and illustrate that the options provide a more flexible way of selling and buying ads. In particular, we show that an advertiser can have better deliveries in a bull market (where the underlying price increases). On the other hand, a publisher or search engine is able to reduce the revenue volatility over time. In a bear market (where the underlying price decreases), there is a growth in total revenue. To our best knowledge, this is the first work that discusses lattice methods for the ad option evaluation.

The rest of the paper is organized as follows. Section 2 reviews the related work. Section 3 introduces the preliminaries of lattice methods for pricing an ad options with the GBM underlying model. Section 4 discusses our lattice method to price an ad option with the SV underlying model. Section 5 presents our experimental results. Section 6 concludes the paper.

2. Literature review

The ad options discussed in this paper are closely connected to financial options, whose evaluation can be traced back to Bachelier (1900), who proposed to use a continuous-time random walk as the underlying process to price an option written on a stock. Samuelson (1965) then replaced the Bachelier's assumption with a geometric form, called the *geometric Brownian motion (GBM)*. Based on the GBM, Black and Scholes (1973) and Merton (1973) discussed a risk-neutral option pricing method independently, called the *Black–Scholes–Merton (BSM) model*, opening the floodgates to option pricing. Various numerical procedures have appeared in this field, including lattice methods, finite difference methods, Monte Carlo simulations, etc. These numerical procedures are capable of evaluating more complex options when the close-form solution does not exist. In our discussion, we focus on lattice methods.

Sharpe (1978) initiated the concept of pricing a call option written on an asset with simple up and down two-state price changes. We call this the *one-step binomial lattice method* and use it as a

pedagogical framework to explain the continuous-time option pricing model without reference to stochastic calculus. Cox et al. (1979) then developed a multi-step binomial framework, called the *Cox–Ross–Rubinstein (CRR) model*, which can converge with the BSM model if the length of the time step is sufficiently small. Boyle (1986) proposed a trinomial lattice, whereby the asset price can either move upwards, downwards, or stay unchanged in a given time period. Other contributors to one factor lattice methods include Kamrad and Ritchken (1991), Tian (1993) and Haahtela (2010). The technical details and differences of these methods are presented in Table 1, where the *movement scale* is the ratio of the price in the next state to the current one, and the *transition probability* is the risk-neutral probability that the asset price moves from the current state to the next one, which is labeled from the upper state to the lower state. It is also worth noting that all of these methods adopt Samuelson's GBM assumption for the underlying asset price.

The GBM assumption may not always be valid empirically. This motivates a general Ornstein–Uhlenbeck (OU) diffusion process for option pricing. Nelson and Ramaswamy (1990) discussed the conditions under which a sequence of binomial processes converges weakly to an OU diffusion process and investigated its application to pricing an option written on an asset with constant volatility. Primbsa et al. (2007) then proposed a pentanomial lattice method that incorporates the skewness and kurtosis of the underlying asset price and found that the limiting distribution is compounded Poisson. Nelson and Ramaswamy (1990) and Primbsa et al. (2007) solved the lattice pricing for the non-GBM underlyings which have constant volatility. Florescu and Viens (2005), Florescu and Viens (2008) proposed lattice methods that deal with a general SV underlying model. However, their method is not very practical in terms of computational efficiency as the transition probabilities are restricted by many conditions and need to be estimated independently before building up the price lattice. From our point of view, a direct censor on transition probabilities of each node, as discussed in Nelson and Ramaswamy (1990), would be more efficient. Our proposed method in Section 4 is based on this idea.

3. Preliminaries of lattice method

This section introduces the basic settings of the lattice based option pricing framework in the context of online advertising. The previous lattice methods introduced in Table 1 are examined. For the reader's convenience, the key notations and terminologies used throughout the paper are described in Table 2. We here discuss the case where an ad option allows its buyer to pay a fixed CPC for display impressions. Therefore, the strike price of the option is the fixed CPC and the underlying price is the uncertain winning payment CPM from RTB, where each single impression being auctioned off is paid at the second highest bid (Edelman et al. 2007, Google 2011). Other ad option cases can be discussed in the same manner, such as an ad option allows its buyer to pay a fixed CPM for display impressions, or an ad option allows its buyer to pay a fixed CPM or CPC for clicks.

Suppose that an advertiser buys an ad option in time 0 which allows him to purchase several impressions from a publisher's ad slot in time 1 at a fixed CPC, denoted by F^C. As impressions are normally auctioned off at a CPM value, the underlying price is the winning payment CPM from RTB, denoted by M_i, $i = 0, 1$. In time 1, the underlying price may rise or fall, denoted by $M_1^{\{u\}}$ or $M_1^{\{d\}}$. Let us consider the upward case. If $M_1^{\{u\}}/(1000H) \geqslant F^C$, the advertiser will exercise the option; if $M_1^{\{u\}}/(1000H) < F^C$, he will not exercise the option but join RTB instead. Note that H represents a constant CTR; therefore, the underlying and strike prices can be compared on the same measurement basis. Mathematically, we

Table 1
Summary of lattice methods used in pricing a call option written on an asset with the GBM underlying model. Detailed description of notations is provided in Table 2.

Model	Movement scales u, d (or u, m, d)	Transition probabilities $q_1, q_2, \cdots q_k$
Binomial lattice (one factor)		
CRR	$u = e^{\sigma\sqrt{\Delta t}}, d = 1/u.$	$q_1 = \frac{e^{r\Delta t}-d}{u-d}, q_2 = 1 - q_1.$
Tian-BIN	$u = \frac{\gamma\zeta}{2}(\zeta + 1 + \sqrt{\zeta^2 + 2\zeta - 3}), \gamma = e^{r\Delta t},$	$q_1 = \frac{e^{r\Delta t}-d}{u-d}, q_2 = 1 - q_1.$
	$d = \frac{\gamma\zeta}{2}(\zeta + 1 - \sqrt{\zeta^2 + 2\zeta - 3}), \zeta = e^{\sigma^2\Delta t}.$	
Haahtela-BIN	$u = e^{\sqrt{e^{\sigma^2\Delta t}-1}+r\Delta t}, d = e^{-\sqrt{e^{\sigma^2\Delta t}-1}+r\Delta t}.$	$q_1 = \frac{e^{r\Delta t}-d}{u-d}, q_2 = 1 - q_1.$
Trinomial lattice (one factor)		
Boyle-TRIN	$u = e^{\lambda\sigma\sqrt{\Delta t}},$	$q_1 = \frac{(\zeta+\gamma^2-\gamma)u-(\gamma-1)}{(u-1)(u^2-1)},$
	$m = 1,$	$q_2 = 1 - q_1 - q_3, \zeta = e^{2r\Delta t}\left(e^{\sigma^2\Delta t} - 1\right),$
	$d = e^{-\lambda\sigma\sqrt{\Delta t}}.$	$q_3 = \frac{(\zeta+\gamma^2-\gamma)u^2-(\gamma-1)u^3}{(u-1)(u^2-1)}, \gamma = e^{r\Delta t}.$
KR-TRIN	$u = e^{\lambda\sigma\sqrt{\Delta t}},$	$q_1 = \frac{1}{2\lambda^2} + \frac{(r-\frac{1}{2}\sigma^2)\sqrt{\Delta t}}{2\lambda\sigma},$
	$m = 1,$	$q_2 = 1 - \frac{1}{\lambda^2},$
	$d = e^{-\lambda\sigma\sqrt{\Delta t}}.$	$q_3 = \frac{1}{2\lambda^2} - \frac{(r-\frac{1}{2}\sigma^2)\sqrt{\Delta t}}{2\lambda\sigma}.$
Tian-TRIN	$u = \varpi + \sqrt{\varpi^2 - m^2},$	$q_1 = \frac{md-\gamma(m+d)+\gamma^2\zeta}{(u-d)(u-m)},$
	$m = \gamma\zeta^2, \gamma = e^{r\Delta t}, \zeta = e^{\sigma^2\Delta t},$	$q_2 = \frac{\gamma(u+d)-ud-\gamma^2\zeta}{(u-m)(m-d)},$
	$d = \varpi - \sqrt{\varpi^2 - m^2}, \varpi = \frac{\gamma}{2}(\zeta^4 + \zeta^3).$	$q_3 = \frac{um-\gamma(u+m)+\gamma^2\zeta}{(u-d)(m-d)}.$

Note: CRR (Cox et al. 1979); Tian-BIN and Tian-TRIN (Tian 1993); Haahtela-BIN (Haahtela 2010); Boyle-TRIN (Boyle 1988); and KR-TRIN (Kamrad and Ritchken 1991).

Table 2
Summary of key notations and abbreviations.

Notation	Description
T	Option expiration date (in terms of year)
n	Total number of time steps and the length of each time step is $\Delta t = T/n$
$\hat{r}, \tilde{r}, r$	Constant risk-less interest rate: $\hat{r}$ is the discrete-time interest rate in Δt; $\tilde{r} = 1 + \hat{r}$; and r is a continuous-time interest rate where $e^{r\Delta t} = \tilde{r}$
u, m, d	State transition size (or movement scale) in upward, unchanged and downward movement
$q_1, \ldots, q_n$	Risk-neutral state transition probability, labeled from the top node to the bottom node
$Q^{(i)}(t_k)$	Risk-neutral probability on node i at time t_k
$\mathbb{Q}$	Risk-neutral probability measure
$\mathbb{P}$	Real-world probability measure
M_i	M_i is CPM at time step i, $i = 0, \ldots, n$
$M(t)$	$M(t)$ is CPM at time t
C_i	C_i is CPC at time step i
$C(t)$	$C(t)$ is CPC at time t
H	Constant CTR
Φ_n	Option payoff on the expiration date
F^M, F^C	Strike price in terms of CPM, CPC
π_0	Option price at time 0 (i.e., the time step 0)
$\mathcal{N}(\cdot)$	Cumulative distribution function of a standard normal distribution
$\mathbf{N}(x, y^2)$	Normal distribution with mean x and standard deviation y, where $x, y \in \mathbb{R}$
μ	Constant drift of the underlying price
σ	Constant volatility of the underlying price
$\sigma(t)$	Stochastic volatility of the underlying price
κ, θ, δ	Constant speed, long-term mean, and volatility of the stochastic volatility model
CPC	Cost-per-click
CPM	Cost-per-mille (i.e., 1000 impressions)
CTR	Click-through rate
$\mathbb{E}[\cdot]$	Expectation
std$(\cdot)$	Standard deviation
$x \wedge y$	$\min\{x, y\}$, where $x, y \in \mathbb{R}$
$(\cdot)^+$	$\max\{0, \cdot\}$

use the option payoff function $\Phi_1^{\{u\}}$ to describe the above decision making, $\Phi_1^{\{u\}} := (M_1^{\{u\}}/(1000H) - F^C)^+$. Similarly, if the winning payment CPM moves downward, the option payoff $\Phi_1^{\{d\}} := (M_1^{\{d\}}/(1000H) - F^C)^+$.

We follow a general economic settings and consider that the advertiser is risk-neutral so that he exercises the ad option only if the option payoff is maximized (Wilmott 2006). We use the so-called risk-neutral probability measure for option pricing (Bjork 2009). In finance, it is defined by the statement that the expected risky return of an asset is equal to a risk-less bank interest return. In the online advertising environment, the risk-neutral probability measure $\mathbb{Q} = (q, 1-q)$ satisfies the following equation

$$\tilde{r}M_0 \equiv quM_0 + (1-q)dM_0, \tag{1}$$

where $\tilde{r} = (1 + \hat{r})$ is the risk-less return over the period from time 0 to time 1, $u = M_1^{\{u\}}/M_0$ and $d = M_1^{\{d\}}/M_0$ are the movement scales of CPM. Therefore, we can obtain the risk-neutral transition probability $q = (\tilde{r} - d)/(u - d)$. Note that here q equals to q_1 in Table 1, which describes the probability that CPM moves upward in time 1. Since the option value can be considered as a bivariate function of time and underlying price, the option value at time 0 can be obtained by discounting the expected option value at time 1 under $\mathbb{Q} = (q, 1-q)$ (Bjork 2009, see Martingale). The option value at time 1 is actually the option payoff; therefore, the option price at time 0 can be obtained by discounting the expected payoff, that is

$$\pi_0 = \tilde{r}^{-1}\mathbb{E}_\mathbb{Q}[\Phi_1] = \tilde{r}^{-1}\left(q\Phi_1^{\{u\}} + (1-q)\Phi_1^{\{d\}}\right). \tag{2}$$

This option price π_0 is fair because it rules out arbitrage (Varian 1987, Bjork 2009). Arbitrage means that an advertiser can obtain a profit larger or smaller than the risk-less bank interest rate with certainty. Consider if the option price is overestimated, i.e., $\pi_0 > \tilde{r}^{-1}(q\Phi_1^{\{u\}} + (1-q)\Phi_1^{\{d\}})$, the advertiser can sell short an ad option at time 0 and save the money into bank to get the risk-less profit $\tilde{r}\pi_0 - (q\Phi_1^{\{u\}} + (1-q)\Phi_1^{\{d\}})$. Converse strategies can be used to obtain arbitrage if the option price is underestimated. Up to this point, we have discussed the option pricing framework that is the one-step binomial method, initially proposed by Sharpe (1978). For a multi-step binomial lattice, as shown in Fig. 1(a), the possible values of CPM and the corresponding risk-neutral transition probabilities can be estimated directly by investigating various combinations of each one-step model, so the option price π_0 can be obtained as follows

$$\pi_0 = \tilde{r}^{-n} \sum_{j=0}^{n} \binom{n}{j} q^j (1-q)^{n-j} \left(\frac{u^j d^{n-j} M_0}{1000H} - F^C \right)^+. \tag{3}$$

If for any $j \geqslant j^*$, $u^j d^{n-j} M_0/(1000H) \geqslant F^C$, then

$$\pi_0 = \frac{M_0}{1000H} \sum_{j=j^*}^{n} \binom{n}{j} \tilde{q}^j (1-\tilde{q})^{n-j} - F^C \tilde{r}^{-n} \sum_{j=j^*}^{n} \binom{n}{j} q^j (1-q)^{n-j}$$

$$= \frac{M_0}{1000H} \psi(j^*, n, \tilde{q}) - F^C \tilde{r}^{-n} \psi(j^*, n, q), \tag{4}$$

where $\tilde{q} = q \times (u/\tilde{r})$. If each time step $\Delta t = T/n$ is sufficiently small, a continuous-time closed-form formula for π_0 can be obtained as follows

$$\pi_0 = \frac{M_0}{1000H} \mathcal{N}(\varsigma_1) - F^C e^{-rT} \mathcal{N}(\varsigma_2), \tag{5}$$

$$\varsigma_1 = \frac{1}{\sigma\sqrt{T}} \left(\ln\left\{ \frac{M_0}{1000HF^C} \right\} + \left(r + \frac{1}{2}\sigma^2 \right) T \right), \tag{6}$$

$$\varsigma_2 = \varsigma_1 - \sigma\sqrt{T}, \tag{7}$$

which is very similar to the BSM model (Black and Scholes 1973, Merton 1973).

Fig. 1(b) exhibits a trinomial lattice. There are 6 parameters: u, m, d are state movement scales; q_1, q_2, q_3 are the corresponding risk-neutral transition probabilities. These parameters uniquely determine the movement of CPM, which then determines a unique value of an ad option written on CPM. They must be restricted such that the constructed trinomial lattice converges to the log-normal distribution of CPM in continuous time (i.e., the GBM assumption). We use the *moment matching technique* (Cox et al. 1979) to define the basic restrictions as follows:

$$q_1 + q_2 + q_3 = 1, \tag{8}$$

$$q_1 u + q_2 m + q_3 d = \gamma = e^{r\Delta t}, \tag{9}$$

$$q_1 u^2 + q_2 m^2 + q_3 d^2 = \gamma^2 \zeta = e^{2r\Delta t} e^{\sigma^2 \Delta t} \tag{10}$$

where $0 \leqslant q_1, q_2, q_3 \leqslant 1$. Since there are 6 parameters, 3 additional equations are necessary to define a unique solution. Here we examine the additional conditions discussed by previous research (Boyle 1988, Kamrad and Ritchken 1991, Tian 1993) and use the same settings to price a display ad option.

Fig. 2 compares the convergence performance of discussed binomial and trinomial lattice methods for option pricing. Eq. (7) is used as the golden line to examine how quickly that the calculated option price from lattice methods approximate to the closed-form value (because these methods are all based on the GBM assumption). Fig. 2(a) illustrates the situation when the option value at time 0 is in the money (i.e., $M_0/(1000H) \geqslant F^C$) and Fig. 2(b) shows the out of the money case (i.e., $M_0/(1000H) < F^C$). Several findings are worth mentioning here. First, the convergence rate of the trinomial lattice is fast than that of the binomial lattice; however, more nodes need to be computed for the former, i.e., $(n+1)^2$ nodes for the trinomial lattice while there are only $(n+1)(n+2)/2$ nodes for binomial lattice. Second, the Tian-TRIN (Tian 1993) model has a better convergence performance than the others.

4. Censored binomial lattice for the SV underlying model

When the GBM assumption is not valid empirically, the SV model can be used to describe the underlying price movement. Let us extend the case whereby an ad option allows its buyer to pay a fixed CPC for display impressions. The SV model for the uncertain winning payment CPM can be expressed as follows:

$$dM(t) = \mu M(t)dt + \sigma(t)M(t)dW(t), \tag{11}$$

$$d\sigma(t) = \kappa(\theta - \sigma(t))dt + \delta\sqrt{\sigma(t)}dZ(t), \tag{12}$$

where $W(t)$ and $Z(t)$ are standard Brownian motions under the real world probability measure $\mathbb{P}$ satisfying $\mathbb{E}[dW(t)dZ(t)] = 0$, and μ and $\sigma(t)$ are the constant drift and volatility of CPM, and κ, θ, δ are the volatility parameters. The drift factor $\kappa(\theta - \sigma(t))$ ensures the mean reversion of $\sigma(t)$ towards its long-term value θ. The volatility factor $\delta\sqrt{\sigma(t)}$ avoids the possibility of negative $\sigma(t)$ for all positive values of κ and θ. It is worth noting that the proposed model is very similar to the Heston model (Heston 1993) while the significant difference is that the hidden layer is driven by $d\sigma(t)$ rather than $d\sigma(t)^2$. Let
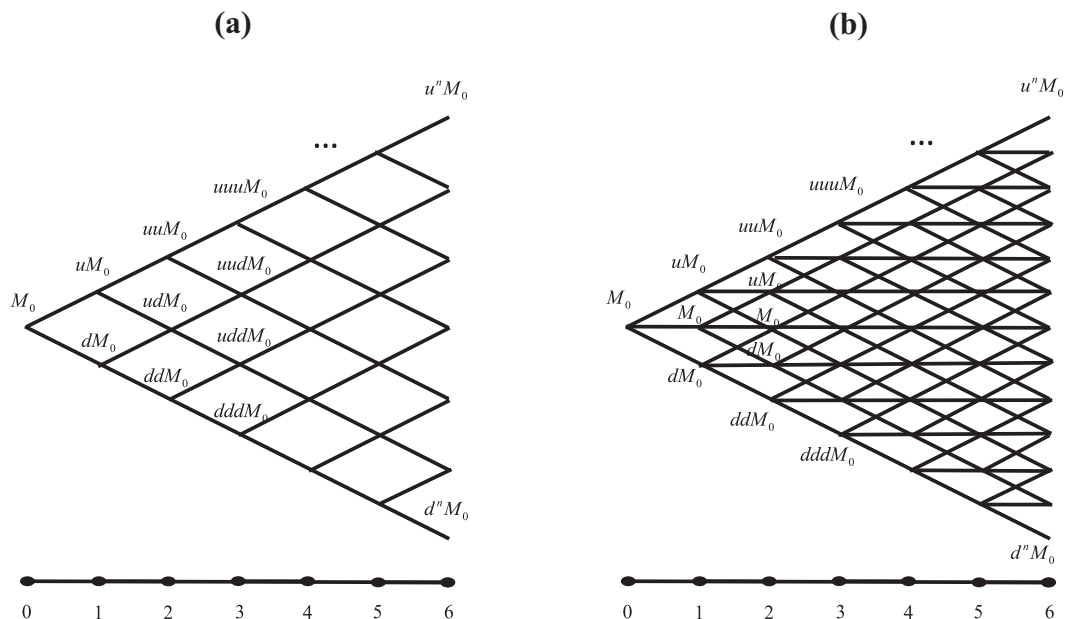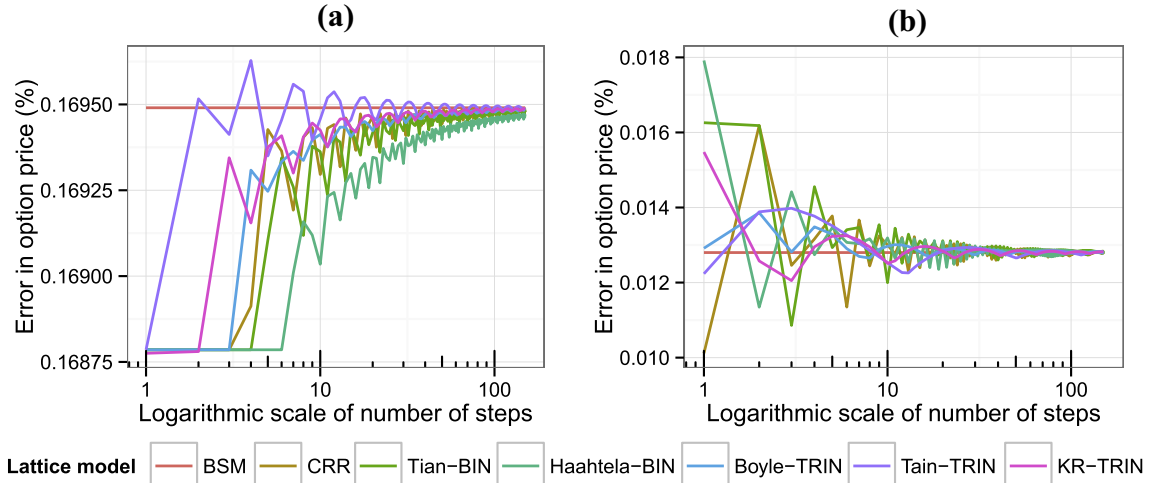
(a)

(b)

Fig. 1. Lattice framework: (a) the binomial lattice for CPM; (b) the trinomial lattice for CPM.

Fig. 2. Comparison of the convergence performance of the binomial and trinomial lattice methods for pricing a display ad option with the GBM underlying: (a) the option value at time 0 is in the money where $M_0 = 2$, $F^C = 0.005$, $CTR = 0.3$, $r = 0.05$, $T = 31/365$ and $\sigma = 0.5$; and (b) the option value at time 0 is out of the money where $M_0 = 2$, $F^C = 0.075$, $CTR = 0.3$, $r = 0.05$, $T = 31/365$ and $\sigma = 0.5$. Detailed descriptions of notations and terminology are provided in Table 2.
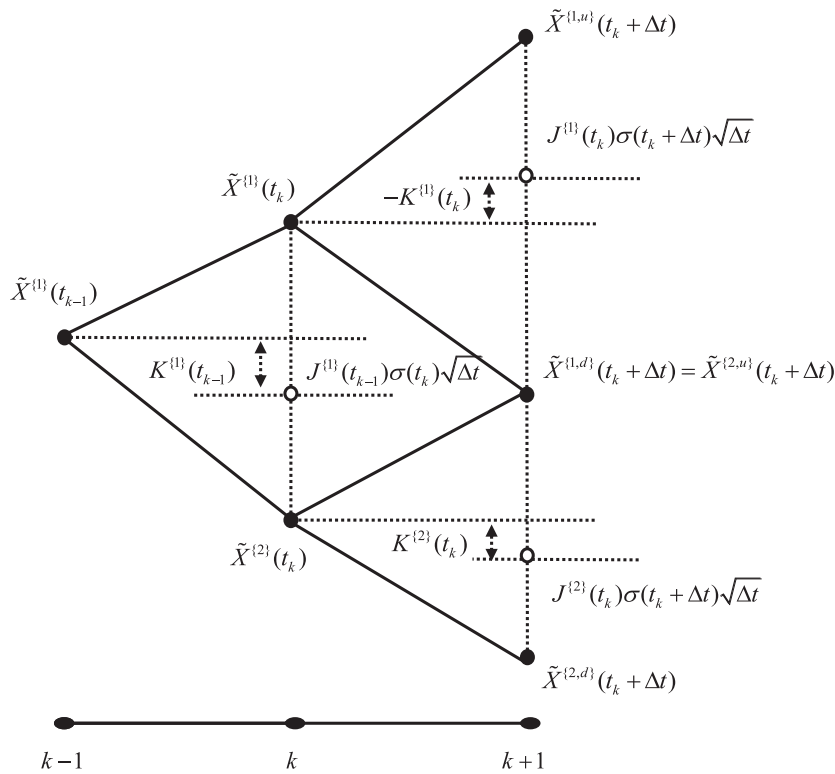
Fig. 3. Censored binomial lattice for the SV underlying. Detailed description of notations is provided in Table 2.

$X(t) = \ln(M(t))$, Eq. (11) can be re-written as the following risk-neutral form:

$$dX(t) = \left(r - \frac{\sigma^2(t)}{2}\right)dt + \sigma(t)dW^{\mathbb{Q}}(t), \tag{13}$$

where r is the constant continuous-time risk-less interest rate and $W^{\mathbb{Q}} := W(t) + \int_0^t \frac{\mu - r}{\sigma(s)}ds$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{Q}$, so $\mathbb{E}[dW^{\mathbb{Q}}(t)dZ(t)] = 0$. The process $X(t)$ can be weakly approximated by a series of binomial processes, say $\widetilde{X}(t_i)$, $i = 1, \ldots, n$. For more details about the

approximation conditions, see (Nelson and Ramaswamy 1990). We will briefly verify these conditions in the following discussion.

In Algorithm 1, we present our method of calculating the option price for a display ad option whose underlying is the SV model. Simply, a binomial lattice for $\widetilde{X}(t_i)$ is first constructed to approximates $X(t)$ weakly. The lattice is constructed from time step 0 to time step n, and at each time step, nodes are calculated from top to bottom. In the following discussion, the mathematical details of Steps ①–③ are introduced.

Step ① We start from the first node $\widetilde{X}^{\{1\}}(t_k)$ in Fig. 3, whose two successors can be expressed as follows

$$\widetilde{X}^{\{1,u\}}(t_k + \Delta t) = (J^{\{1\}}(t_k) + 1)\sigma(t_k + \Delta t)\sqrt{\Delta t}$$
$$+ \left(r - \frac{\sigma^2(t_k + \Delta t)}{2}\right)\Delta t, \tag{14}$$

$$\widetilde{X}^{\{1,d\}}(t_k + \Delta t) = (J^{\{1\}}(t_k) - 1)\sigma(t_k + \Delta t)\sqrt{\Delta t}$$
$$+ \left(r - \frac{\sigma^2(t_k + \Delta t)}{2}\right)\Delta t, \tag{15}$$

where $J^{\{1\}}(t_k)\sigma(t_k + \Delta t)\sqrt{\Delta t}$ is the point on the grid closest to $\widetilde{X}^{\{1\}}(t_k)$, given by

$$J^{\{1\}}(t_k) = \inf_{J^* \in \mathbb{N}}\left|J^*\sigma(t_k + \Delta t)\sqrt{\Delta t} - \widetilde{X}^{\{1\}}(t_k)\right|. \tag{16}$$

Eqs. (14) and (15) can be rewritten in terms of their conditional increments:

$$\widetilde{X}^{\{1,u\}}(t_k + \Delta t) - \widetilde{X}^{\{1\}}(t_k) = \sigma(t_k + \Delta t)\sqrt{\Delta t} - K^{\{1\}}(t_k)$$
$$+ \left(r - \frac{\sigma^2(t_k + \Delta t)}{2}\right)\Delta t, \tag{17}$$

$$\widetilde{X}^{\{1,d\}}(t_k + \Delta t) - \widetilde{X}^{\{1\}}(t_k) = -\sigma(t_k + \Delta t)\sqrt{\Delta t} - K^{\{1\}}(t_k)$$
$$+ \left(r - \frac{\sigma^2(t_k + \Delta t)}{2}\right)\Delta t, \tag{18}$$

where $K^{\{1\}}(t_k)$ is the grid adjusting parameter for the successors of the first node at time t_k. As shown in Fig. 3, the value of $K^{\{i\}}(t_k)$, $i = 1, 2, \ldots, k$, can be either positive or negative, To satisfy the approximation condition $\lim_{\Delta t \to 0}|X(t_k + \Delta t) - X(t_k)| = 0$, the following equation holds:

$$\mathbb{E}\left[\widetilde{X}^{\{1\}}(t_k + \Delta t) - \widetilde{X}^{\{1\}}(t_k)|\mathcal{F}(t_k)\right] = \left(r - \frac{\sigma^2(t_k + \Delta t)}{2}\right)\Delta t. \tag{19}$$

Then, we can obtain a system of equations

$$\left(\sigma(t_k + \Delta t)\sqrt{\Delta t} - K^{\{1\}}(t_k)\right)\frac{q_1^{\{1\}}(t_k)}{Q^{\{1\}}(t_k)}$$
$$+ \left(-\sigma(t_k + \Delta t)\sqrt{\Delta t} - K^{\{1\}}(t_k)\right)\frac{q_2^{\{1\}}(t)}{Q^{\{1\}}(t)} = 0,$$
$$q_1^{\{1\}}(t_k) + q_2^{\{1\}}(t_k) = Q^{\{1\}}(t),$$

where $q_1^{\{1\}}(t_k)$ and $q_2^{\{1\}}(t_k)$ are the risk-neutral probabilities that the successor of the first node at time t_k rises or falls in time $t_k + \Delta t$, and $Q^{\{1\}}(t_k)$ is the risk-neutral probability for the first node at time t_k. Solving the above equations then gives

$$q_1^{\{1\}}(t_k) = \begin{cases} \frac{Q^{\{1\}}(t_k)}{2}\left(1 + \frac{K^{\{1\}}(t_k)}{\sigma(t_k + \Delta t)\sqrt{\Delta t}}\right), \\ \quad \text{if } 0 \leqslant \frac{Q^{\{1\}}(t_k)}{2}\left(1 + \frac{K^{\{1\}}(t_k)}{\sigma(t_k + \Delta t)\sqrt{\Delta t}}\right) \leqslant Q^{\{1\}}(t_k), \\ 0, \text{if } \frac{Q^{\{1\}}(t_k)}{2}\left(1 + \frac{K^{\{1\}}(t_k)}{\sigma(t_k + \Delta t)\sqrt{\Delta t}}\right) < 0, \\ Q^{\{1\}}(t_k), \text{if } \frac{Q^{\{1\}}(t_k)}{2}\left(1 + \frac{K^{\{1\}}(t_k)}{\sigma(t_k + \Delta t)\sqrt{\Delta t}}\right) \geqslant Q^{\{1\}}(t_k), \end{cases} \tag{20}$$

$$= \left(Q^{\{1\}}(t_k) \wedge \frac{Q^{\{1\}}(t_k)}{2}\left(1 + \frac{K^{\{1\}}(t_k)}{\sigma(t_k + \Delta t)\sqrt{\Delta t}}\right)\right)^+,$$

$$q_2^{\{1\}}(t_k) = Q^{\{1\}}(t_k) - q_1^{\{1\}}(t_k). \tag{21}$$

Algorithm 1. Censored binomial lattice method for pricing a display ad option with the SV underlying model. Detailed description of notations is provided in Table 2.

---

function
  `OptionPricingBinSV`$(M_0, \sigma_0, \kappa, \theta, \delta, H, T, n, r, F^C)$
  $\Delta t \leftarrow T/n; \widetilde{r} \leftarrow e^{r\Delta t};$
  for $k \leftarrow 0$ to $n - 1$ **do**
    for $i \in$ nodes in time step k **do**
      if $i = 1$ **then**
        Step ①;
      else
        Step ②;
      end if
    end for
  end for
  $\pi_0 \leftarrow$ Eq. (22) (see Step ③);
end function

Eqs. (20) and (21) show that transition probabilities $q_1^{\{1\}}(t_k)$ and $q_2^{\{1\}}(t_k)$ are censored in the approximation.

Step ② The successors of other nodes can be constructed in the same manner as that of $\widetilde{X}^{\{1\}}(t_k)$. Since the transition probabilities are censored directly at each node, $K^{\{i\}}(t_k)$, $J^{\{i\}}(t_k)$ and $Q^{\{i\}}(t_k)$ can be calculated sequentially from top to bottom alongside the lattice construction for the underlying price. The nodes need to be kept the recombining pattern; therefore, the following equations hold for $1 \leqslant i \leqslant k$:

$$\widetilde{X}^{\{i,d\}}(t_k + \Delta t)$$
$$= (J^{\{i\}}(t_k) - 1)\sigma(t_k + \Delta t)\sqrt{\Delta t} + \left(r - \frac{\sigma^2(t_k + \Delta t)}{2}\right)\Delta t$$
$$= \widetilde{X}^{\{i+1,u\}}(t_k + \Delta t)$$
$$= (J^{\{i+1\}}(t_k) + 1)\sigma(t_k + \Delta t)\sqrt{\Delta t} + \left(r - \frac{\sigma^2(t_k + \Delta t)}{2}\right)\Delta t,$$

therefore

$$J^{\{i+1\}}(t_k) = J^{\{i\}}(t_k) - 2,$$
$$K^{\{i+1\}}(t_k) = J^{\{i+1\}}(t_k)\sigma(t_k + \Delta t)\sqrt{\Delta t} - \widetilde{X}^{\{i+1\}}(t_k).$$

The transition probabilities for the node $\widetilde{X}^{\{i+1\}}(t_k)$ can be then estimated by Eqs. (20) and (21). Hence, the rolling risk-neutral probability distribution $Q^{\{i\}}(t_k)$ for each node can be quickly computed as follows:

$$Q^{\{i\}}(t_k + \Delta t) = \begin{cases} q_1^{\{1\}}(t_k), & \text{if } i = 1, \\ q_2^{\{i-1\}}(t_k) + q_1^{\{i\}}(t_k), & \text{if } 1 < i < k + 1, \\ q_2^{\{k+1\}}(t_k), & \text{if } i = k + 1, \end{cases}$$

subjected to the initial condition $Q(t_0) = 1$.

Step ③ The binomial lattice can be constructed by steps ①–② for each time step until the contract expiration date. Finally, the option price can be obtained as follows:

$$\pi_0 = \widetilde{r}^{-n}\sum_{i=1}^{n+1} Q^{\{i\}}(t_n)\left(\frac{1}{1000H}e^{\widetilde{X}^{\{i\}}(t_n)} - F^C\right)^+. \tag{22}$$

Similar to Eq. (4), Eq. (22) is also the discrete form of the risk-neutral terminal pricing (Bjork 2009).

In the above discussion, we actually followed Florescu and Viens (2005) to construct the binomial lattice and used variables $K^{\{i\}}(t_k)$ and $J^{\{i\}}(t_k)$ to tune the grid so that the constructed binomial
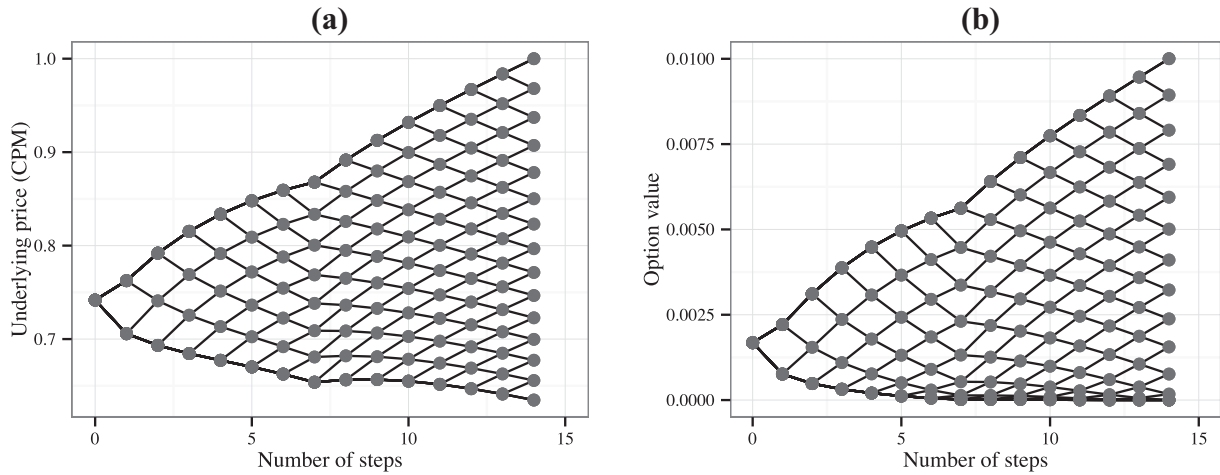
Fig. 4. Empirical example of binomial lattices for an ad slot from the SSP dataset: (a) the censored binomial lattice for CPM based on the SV model, where $r = 0.05$, $T = 0.0384$, $n = 14$, $CPM = 0.7417$, $\sigma_0 = 0.8723$, $\kappa = 96.4953$, $\theta = 0.2959$, $\delta = 14.9874$; (b) the censored binomial lattice for the option value. The model parameters are estimated based on the training data.
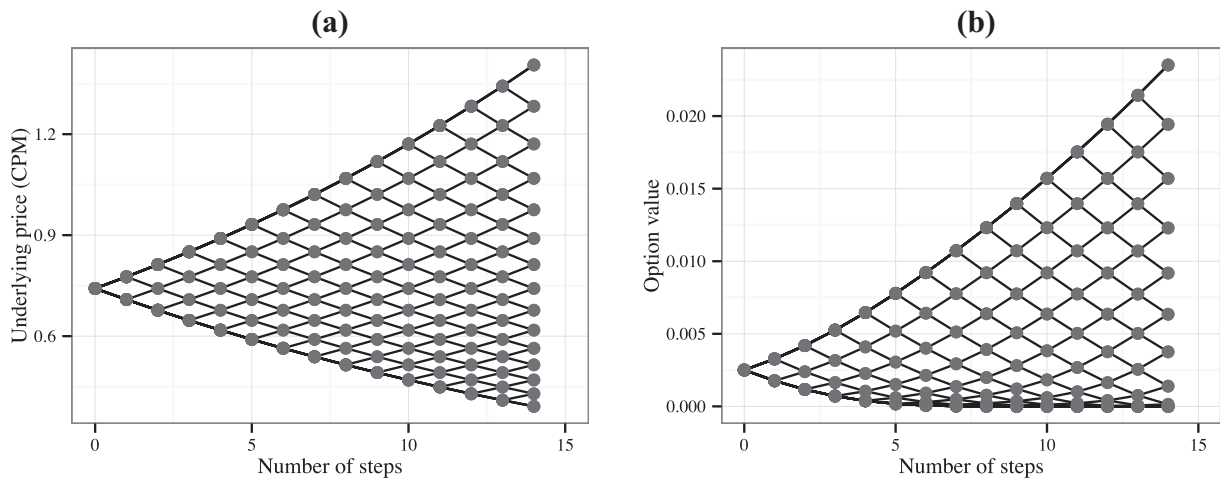
Fig. 5. Example of binomial lattices for the same ad slot in Fig. 4: (a) the CRR binomial lattice for CPM based on the GBM model, where $r = 0.05$, $T = 0.0384$, $n = 14$, $CPM = 0.7417$, $\sigma_0 = 0.8723$. Here we use the same parameters' values in Fig. 4; (b) the CRR binomial lattice for the option value.

Table 3
Summary of datasets for experiments.

Dataset	SSP	Google AdWords
Period	08/01/2013–14/02/2013	26/11/2011–14/01/2013
No. of ad slots or keywords	31	557
No. of advertisers	374	×
No. of impressions	6646643	×
No. of bids	33043127	×
Winning payment price	√	√
Bid quote	GBP/CPM	GBP/CPC

Table 4
Experimental settings of the SSP dataset.

Training set (31 days)	Development & test set (7 days)
08/01/2013–07/02/2013	08/02/2013–14/02/2013

framework is recombining. Compared to Florescu and Viens (2005), our method simplifies the lattice construction process by censoring the probabilities at each node directly. In the meantime,

the structure satisfies the approximation conditions proposed by Nelson and Ramaswamy (1990). Fig. 4 presents an empirical example of constructing a censored binomial lattice for pricing a display ad option written on an ad slot from a SSP in the UK. The given values of the model parameters are estimated from the training data. Fig. 4(a) shows a censored binomial lattice for the underlying CPM and Fig. 4(b) illustrates how the option value is calculated backward iteratively from the expiration date to time 0. For the sake of comparison, Fig. 5 illustrates the binomial lattices constructed by the CRR model with the same parameter settings. Obviously, the changing volatility can be found in Fig. 4(a) while 5(a) exhibits a constant volatility over time. We find that the option price given by the SV model is slightly smaller than that of the CRR model. This is because the long-term mean value of volatility is 0.2959, smaller than its initial value 0.8723. Therefore, the drift drags the volatility downside to its long-term level and the option value based on the SV model contains less risk than the CRR model.

5. Empirical evaluation

This section presents our experimental results. We examine the GBM assumption with real advertising data, compare the fitness of

Table 5
Experimental settings of the Google AdWords dataset.

Market	Group	Training set (31 days)	Development & test set (31 days)
US	1	25/01/2012–24/02/2012	24/02/2012–25/03/2012
	2	30/03/2012–29/04/2012	29/04/2012–31/05/2012
	3	10/06/2012–12/07/2012	12/07/2012–17/08/2012
	4	10/11/2012–11/12/2012	11/12/2012–10/01/2013
UK	1	25/01/2012–24/02/2012	24/02/2012–25/03/2012
	2	30/03/2012–29/04/2012	29/04/2012–31/05/2012
	3	12/06/2012–13/07/2012	13/07/2012–19/08/2012
	4	18/10/2012–22/11/2012	22/11/2012–24/12/2012
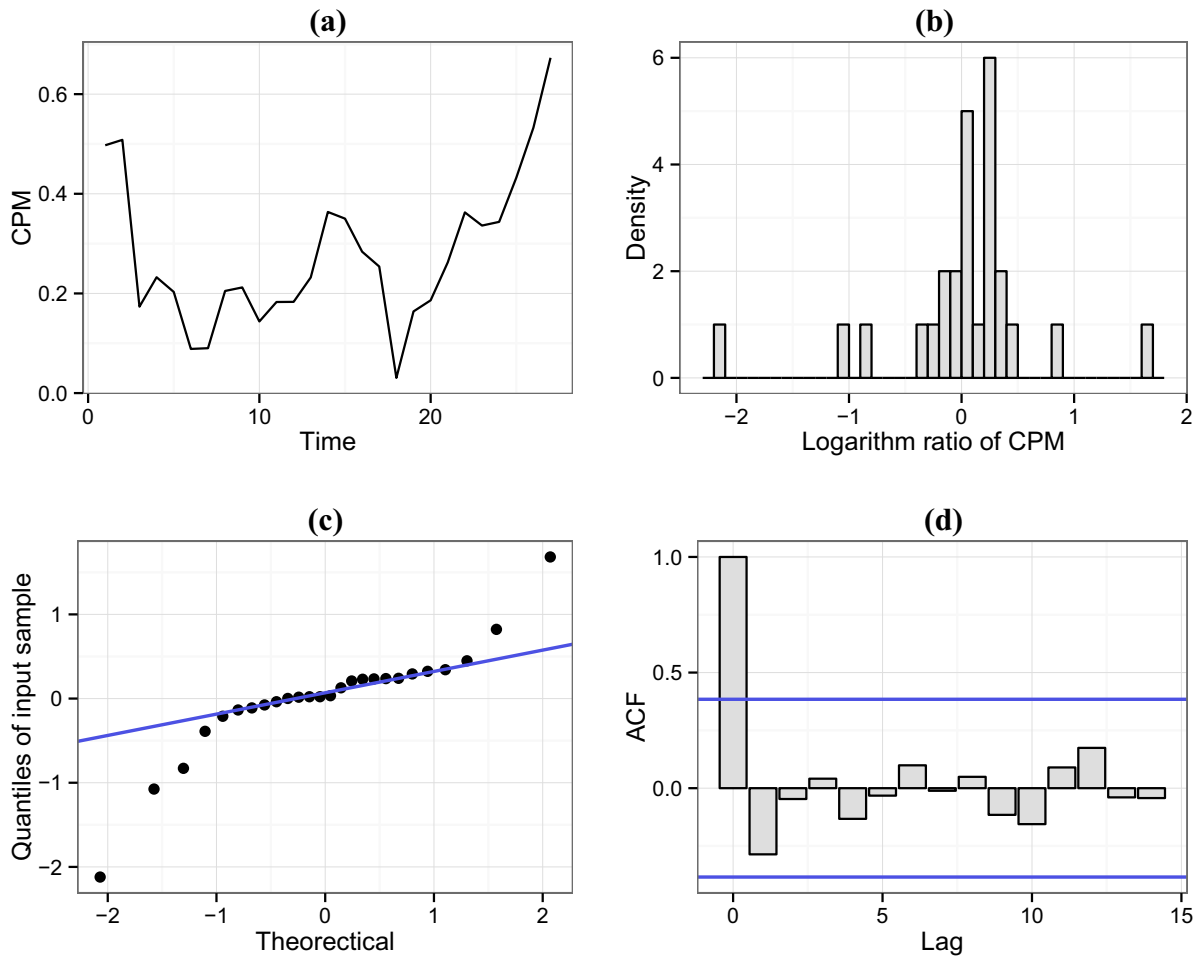
Fig. 6. Empirical example of testing the GBM conditions of an ad slot from the SSP dataset: (a) the plot of the average daily winning payment CPMs from auctions; (b) the histogram of the logarithm ratios of the CPM, i.e., $\ln(M_{i+1}/M_i)$, $i = 1, \ldots, n-1$; (c) the QQ plot of the logarithm ratios; (d) the plot of the ACFs of the logarithm ratios. The Shapiro–Wilk test is with p-value 0.0009 and the Ljung–Box test is with p-value 0.1225.

underlying models, validate the proposed lattice method via Monte Carlo simulations, analyze if an advertiser can have better deliveries under a fixed daily budget, and discuss the effects on the publisher's (or search engine's) revenue.

5.1. Datasets and experimental design

Table 3 presents the two datasets used in experiments: a RTB dataset from a SSP in the UK; and a sponsored search dataset from Google AdWords. The RTB dataset contains all advertisers' bids and the corresponding winning payment CPMs (per transaction). The Google dataset is obtained by using Google's Traffic Estimation service Yuan and Wang 2012. Tables 4 and 5 illustrates our experimental settings. Each dataset is divided into several experimental groups and each group is specified with one training, one development and one test set. The model parameters are estimated in the training set. Display ad options are priced in the development set. The actual bids in the test set are used to examine the priced options. The default value of CTR is set to be 0.03.

5.2. Fitness of GBM and SV models

The following two conditions hold if the GBM assumption is valid empirically: (i) the normality of the logarithm ratios of the winning payment price[1]; and (ii) the independence of the logarithm

---
[1] The logarithm ratio of winning payment price L_i is defined by $L_i = \ln(M_{i+1}/M_i)$ or $L_i = \ln(C_{i+1}/C_i)$.
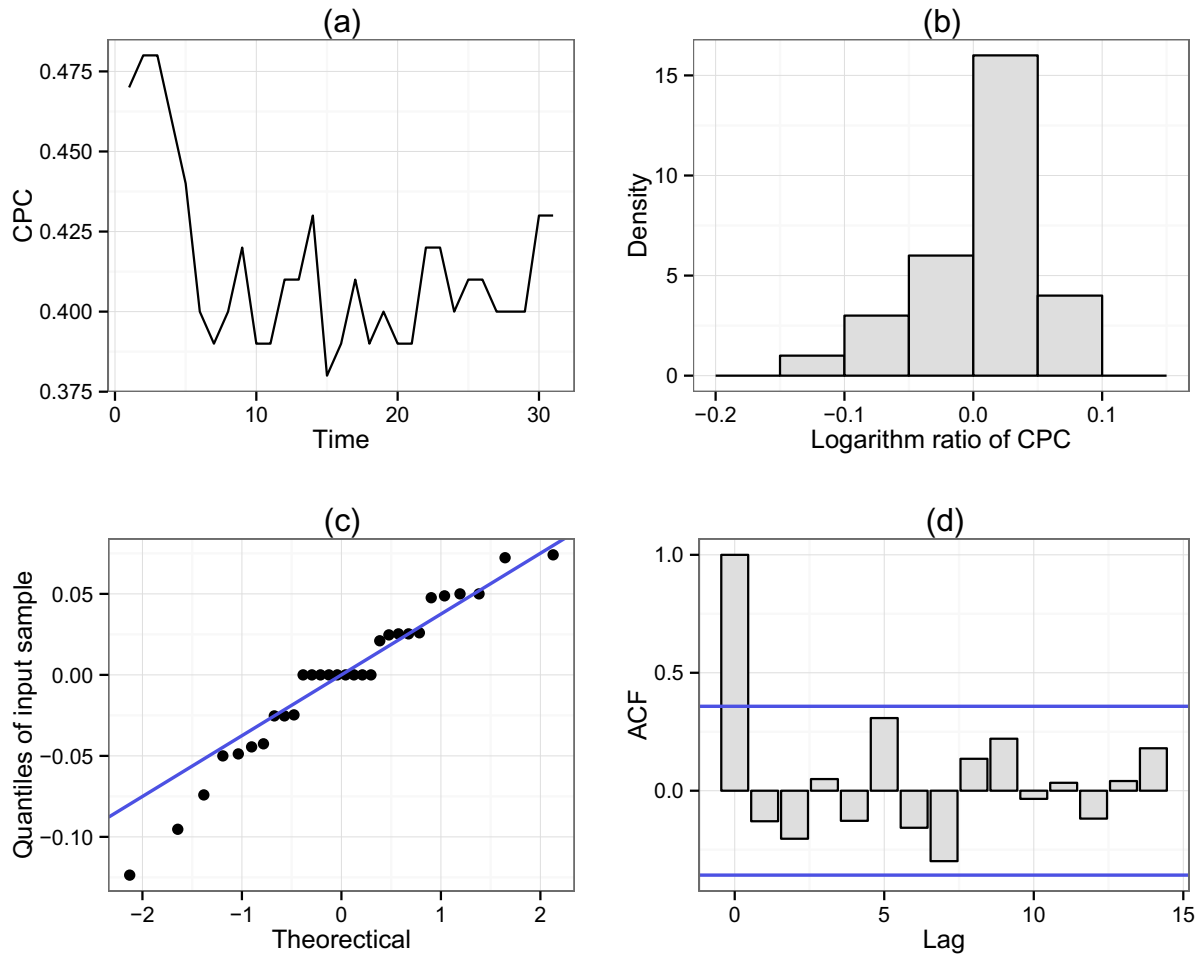
Fig. 7. Empirical example of testing the GBM conditions of the keyword "canon 5d" from the Google AdWords dataset: (a) the plot of average daily winning payment CPCs; (b) the histogram of logarithm ratios of CPC, i.e., $\ln(C_{i+1}/C_i)$, $i = 1, \dots, n-1$; (c) the QQ plot of the logarithm ratios; (d) the plot of the ACFs of the logarithm ratios. The Shapiro–Wilk test is with p-value 0.2144 and the Ljung–Box test is with p-value 0.6971.
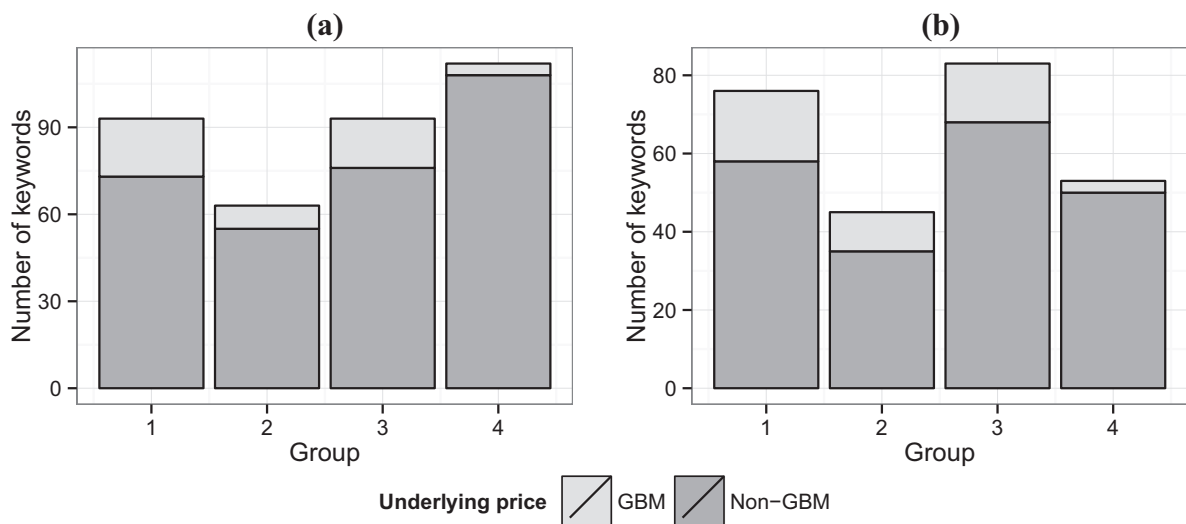
Fig. 8. Summary of the GBM conditions test for all keywords in the Google AdWords dataset.

ratios from the previous data. Normality can be graphically checked by a histogram or Q–Q plot, and be statistically verified by the Shapiro–Wilk test (Shapiro and Wilk 1965); independence can be tested by the autocorrelation function (ACF) (Tsay 2005) and the Lj

ung–Box statistic (Ljung and Box 1978). It is worth noting that the above two conditions are necessary conditions while we follow Marathe and Ryan (2005) and consider the GBM assumption is valid empirically if they are not rejected by real data.
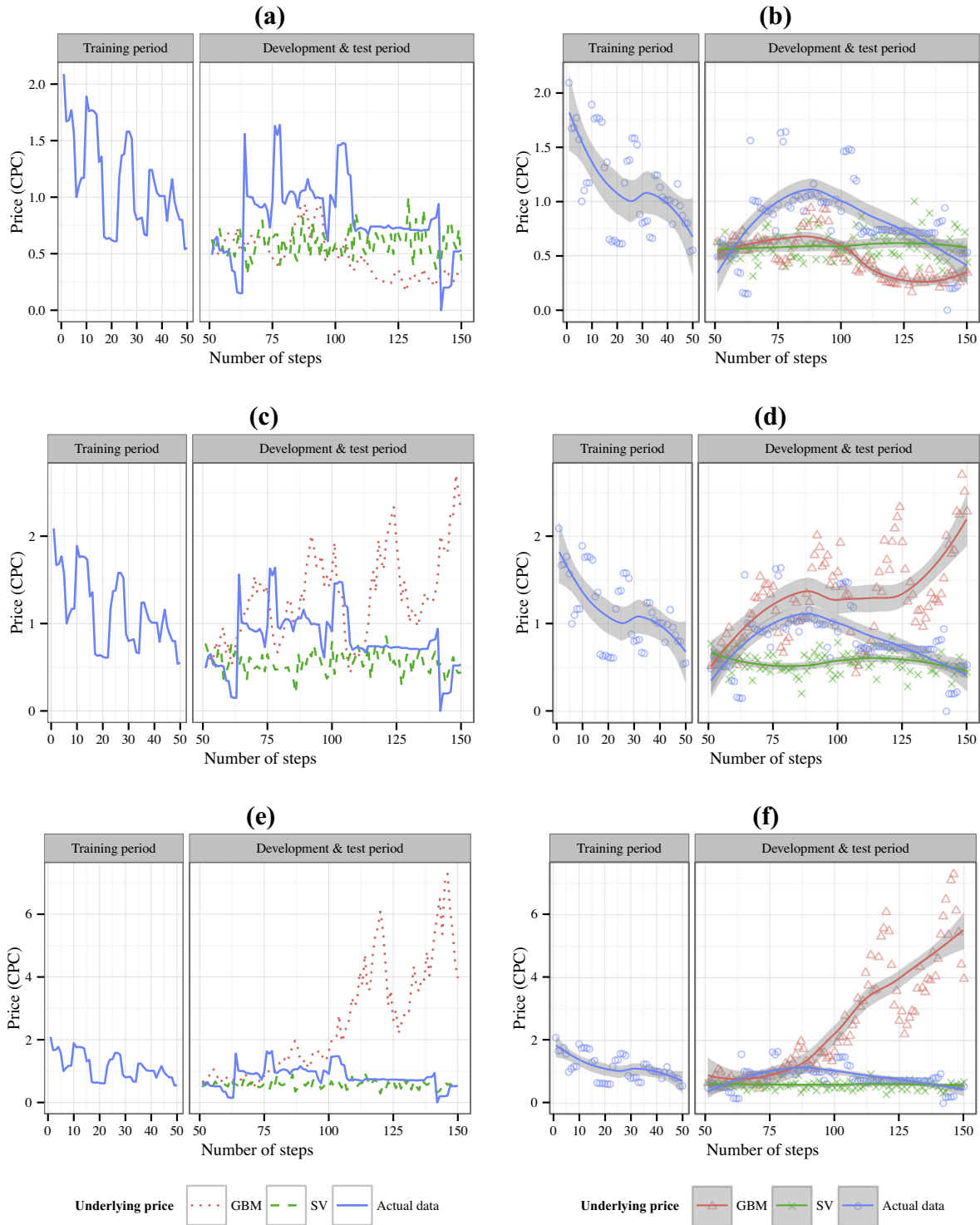
Fig. 9. Empirical example of comparing the fitness of GBM and SV models for the keyword "kinect for xbox 360" from the Google AdWords dataset. The training period is from time steps 1–50, the development and test periods are from time steps 51–150. Plot (a), (c), (e) illustrate three instances of simulated paths from the estimated GBM and SV, respectively. Plot (b), (d), (f) provide the corresponding smooth pattern and confidence interval of plot (a), (c), (e).

Fig. 6 presents an empirical example of testing the GBM assumption for an ad slot from the SSP dataset, where the underlying winning CPM cannot be described accurately as a GBM. In fact, none of the 31 ad slots in the SSP dataset satisfy the GBM model. Therefore, we use the SV model for the ad slots in the SSP dataset.

Fig. 7 presents an example of a keyword from the Google dataset. The keyword's winning CPC satisfies the GBM assumption. The log-normality of CPC is validated in Fig. 7(a)–(c) and the independence is confirmed by Fig. 7(d). The overview results of the Google dataset is shown in Fig. 8. There are 14.25% and 17.20% of the

keywords in the US and UK markets respectively that can be accurately described by the GBM model. We will price the remaining keywords using the SV model.

Fig. 9 gives an empirical example showing the model fitness for the situation where the GBM assumption is not valid. Three different instances of simulated paths are generated from the GBM and SV models for the same keyword. Fig. 9(a), (c), (d) compares the simulations from these two models with the actual winning payment CPCs in real-time auctions. The smooth movement pattern of these three instances are also examined in Fig. 9(b), (d), (f). It is obvious that the SV model has a better fitness to the data. In addition, the Euclidean distance (also called the *L-2 distance*) is used to examine the similarity of the simulated path and the test data. The overall results of the ad slots and keywords in our datasets are presented in Tables 6 and 7, which show that the SV model has a general better fitness to real data.

5.3. Validation of the option pricing model

We now examine the proposed ad option pricing method via two sequential Monte Carlo simulation methods. By using the terminal value pricing formula (Bjork 2009), the option price π_0 can be estimated as follows:

$$\pi_0 = \sum_{j=1}^{\tilde{n}} \tilde{r}^{-n} \left(\frac{1}{1000H} M_j(t_n) - F^C \right)^+ , \tag{23}$$

where $M_j(t_n)$ can be generated by either Euler or Milstein discretisation schemes (Glasserman 2003):

Euler Scheme

$$M(t_i + \Delta t) = M(t_i) e^{(r - \frac{1}{2}\sigma^2(t_i))\Delta t + \sigma(t_i)\sqrt{\Delta t}\epsilon_i}, \tag{24}$$

$$\sigma(t_i + \Delta t) = \sigma(t_i) + \kappa(\theta - \sigma(t_i))\Delta t + \delta\sqrt{\sigma(t_i)\Delta t}\varepsilon_i, \tag{25}$$

Milstein Scheme

$$M(t_i + \Delta t) = M(t_i) e^{(r - \frac{1}{2}\sigma^2(t_i))\Delta t + \sigma(t_i)\sqrt{\Delta t}\epsilon_i}, \tag{26}$$

$$\sigma(t_i + \Delta t) = \sigma(t_i) + \kappa(\theta - \sigma(t_i))\Delta t + \delta\sqrt{\sigma(t_i)\Delta t}\varepsilon_i + \frac{1}{4}\delta^2\Delta t(\varepsilon_i^2 - 1), \tag{27}$$

where $\epsilon_i \sim \mathbf{N}(0,1)$, $\varepsilon_i \sim \mathbf{N}(0,1)$.

These two methods have been widely used in validating the pricing models for exotic options in finance. There are two strong benefits. First, they are developed directly based on the discretisation forms of the underlying dynamics, easy to implement and have good convergence performance to the closed-form solution. Second, they provide a natural criteria for controlling errors. Consider that the errors are controlled with 95% probability, the following criteria can be used to test the option price calculated from our proposed model:

$$\pi_0^{\mathbf{BinSV}} \in \left[\underbrace{\pi_0^{\mathbf{MC}} - 1.96\frac{\tilde{r}^{-n}\mathrm{std}(\Phi(M(t_n)))}{\sqrt{\tilde{n}}}}_{=\pi_0^{\mathbf{MC.Lower}}}, \underbrace{\pi_0^{\mathbf{MC}} + 1.96\frac{\tilde{r}^{-n}\mathrm{std}(\Phi(M(t_n)))}{\sqrt{\tilde{n}}}}_{=\pi_0^{\mathbf{MC.Upper}}} \right],$$

where $\pi_0^{\mathbf{BinSV}}$ represents the option price calculated from our proposed censored binomial lattice method, $\pi_0^{\mathbf{MC}}$ represents the option price calculated from Monte Carlo simulations, $\pi_0^{\mathbf{MC.Lower}}$ and $\pi_0^{\mathbf{MC.Upper}}$ represent the lower and upper bounds of $\pi_0^{\mathbf{MC}}$, respectively.

Fig. 10 provides our model validation test. We price an ad option using the proposed censored binomial lattice and the discussed two Monte Carlo simulation methods. The model parameters are changed in certain intervals against each other in order to investigate the sensitivity of the calculated option price to the values of parameters. It is not difficult to see that the proposed lattice method is robust and accurate because $\pi_0^{\mathbf{BinSV}}$ is very close to $\pi_0^{\mathbf{MC}}$ and always lies in the confidence interval for different model parameters' values.

5.4. Delivery performance for advertiser

Tables 8 and 9 present an empirical example that compares an advertiser's delivery performance between RTB and ad options. Table 8 shows the advertiser's delivery performance in RTB with a fixed daily budget. If the supplied impressions are at same levels and if the average winning payment CPMs increase, the advertiser will receive fewer impressions. In Table 9, the advertiser buys several ad options in advance. Consider if he purchases an ad option with expiration date 08/02/2013, he has the right to secure impressions that will be created on 08/02/2013 at a fixed CPC. Here the advertiser is assumed to use his daily budget from the corresponding delivery date to pay the upfront option price. Hence, as shown in Table 9, the advertiser's strategy is to purchase as many options as possible, and the remaining daily budgets will be used
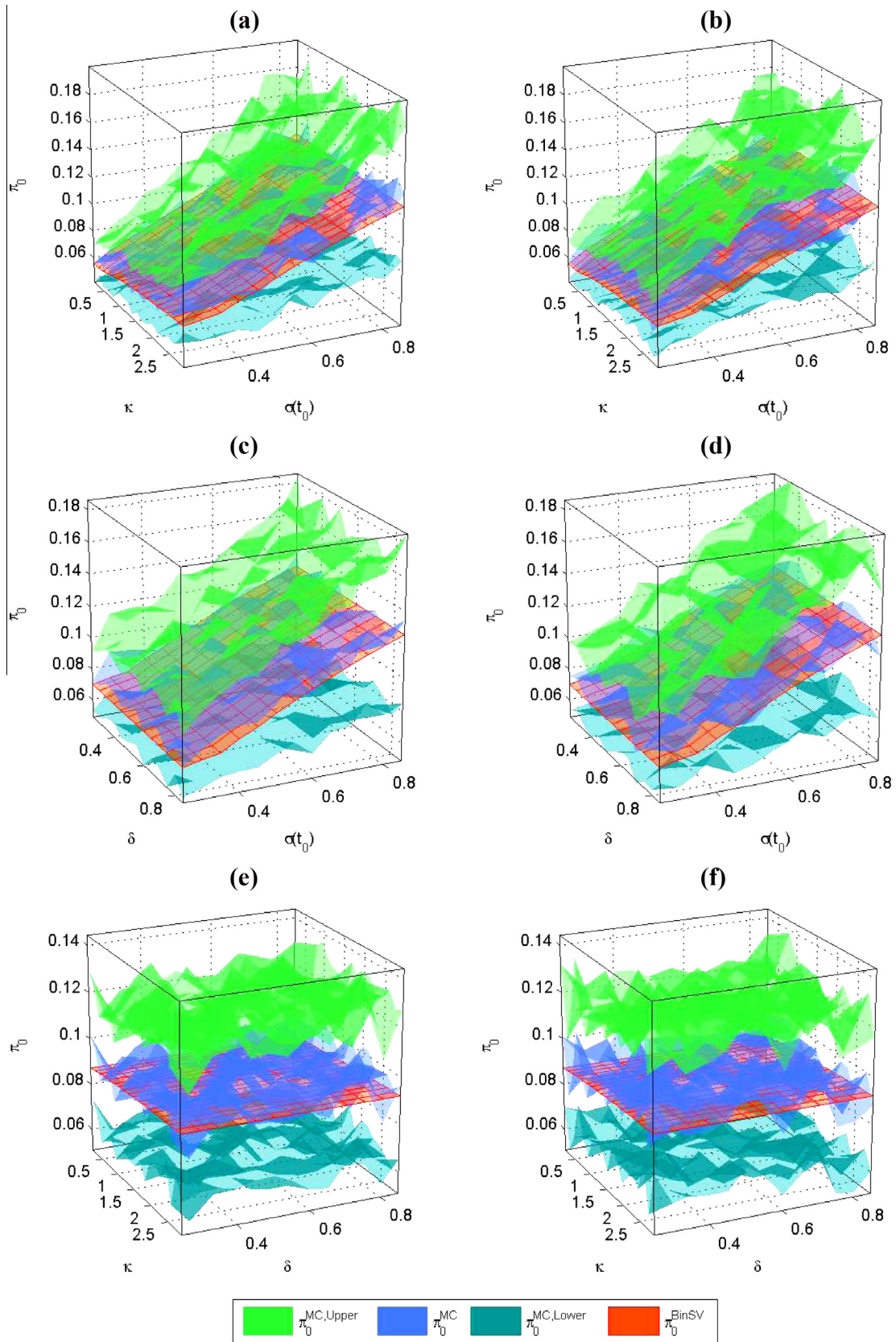
Table 6
Comparing the model fitness for all 31 ad slots in the SSP dataset. L-2 distance is the Euclidean distance, and the number represents the percentage of ad slots which shows that the SV model has a better fitness (i.e., a smaller L-2 distance).

Training set (31 days)	Development & test set (7 days)	L2 distance of simulated paths	L2 distance of smoothed simulated paths
08/01/2013–07/02/2013	08/02/2013–14/02/2013	54.8387%	67.7419%

Table 7
Comparing the model fitness for the non-GBM keywords in the Google AdWords dataset. L-2 distance is the Euclidean distance, and the number represents the percentage of non-GBM keywords which shows that the SV model has a better fitness (i.e., a smaller L-2 distance).

Market	Group	Training set	Development & test set (31 days)	L2 distance of simulated paths (%)	L2 distance of smoothed simulated paths (%)
US	1	25/01/2012–24/02/2012	24/02/2012–25/03/2012	82.8571	80.0000
	2	30/03/2012–29/04/2012	29/04/2012–31/05/2012	94.8718	96.1538
	3	10/06/2012–12/07/2012	12/07/2012–17/08/2012	64.2857	64.2857
	4	10/11/2012–11/12/2012	11/12/2012–10/01/2013	98.1481	100.0000
UK	1	25/01/2012–24/02/2012	24/02/2012–25/03/2012	96.3636	90.9091
	2	30/03/2012–29/04/2012	29/04/2012–31/05/2012	98.2456	94.7368
	3	12/06/2012–13/07/2012	13/07/2012–19/08/2012	58.0645	67.7419
	4	18/10/2012–22/11/2012	22/11/2012–24/12/2012	72.2222	80.5556

Fig. 10. Example of model validation tests: (a), (c), (e) Euler scheme; (b), (d), (f) Milstein scheme. The initial values and parameters settings are: $M(t_0) = 20$, $F^C = 0.633$, $r = 0.05$, $\sigma(t_0) = 0.5$, $\kappa = 3$, $\theta = 0.75$, $\delta = 0.35$.

Table 8
Empirical example of an advertiser's delivery of an ad slot from the SSP dataset in RTB (Note: CTR is 0.03 and the non-integer numbers are displayed at 4 digits after the decimal point while in computing we consider 25-digit scale).

Day	0	1	2	3	4	5	6	7	Total
Date	07/02/2013	08/02/2013	09/02/2013	10/02/2013	11/02/2013	12/02/2013	13/02/2013	14/02/2013	
Average payment CPM	0.7427	0.9585	0.9770	0.9666	0.8754	0.8513	0.8294	0.9903	
No. of total impressions generated		8298	8277	8190	7971	8097	8201	3812	52846
Budget		5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	35.0000
No. of impressions received		5210	5113	5166	5711	5867	6028	3812	36907
No. of clicks received		156	153	154	171	176	180	114	1104
Used budget		5.0000	5.0000	5.0000	4.9830	5.0000	4.9926	3.7748	33.7504

Table 9
Empirical example of an advertiser's delivery of buying ad options for an advertisement slot in the SSP dataset (Note: CTR is 0.03 and the non-integer numbers are displayed at 4 digits after the decimal point while in computing we consider 25-digit scale).

Day	Date	Average payment CPM	No. of total impressions generated	Budget	Remaining budget	No. of options	Expiration date	Option price	Strike price CPC	No. of options exercised	No. of impressions received	No. of clicks received	Used budget
0	07/02/2013	0.7427				201	08/02/2013	0.0025	0.0223				0.4982
		0.7427				201	09/02/2013	0.0025	0.0223				0.4988
		0.7427				201	10/02/2013	0.0025	0.0223				0.5117
		0.7427				201	11/02/2013	0.0026	0.0223				0.5192
		0.7427				200	12/02/2013	0.0026	0.0223				0.5271
		0.7427				200	13/02/2013	0.0027	0.0223				0.5379
		0.7427				199	14/02/2013	0.0027	0.0223				0.5427
1	08/02/2013	0.9585	8298	5.0000	4.5018	201	08/02/2013	0.0022	0.0223	201	6816	204	4.5013
2	09/02/2013	0.9770	8277	5.0000	4.5012	201	09/02/2013	0.0021	0.0223	201	6770	203	4.5011
3	10/02/2013	0.9666	8190	5.0000	4.4883	201	10/02/2013	0.0019	0.0223	201	6742	202	4.4878
4	11/02/2013	0.8754	7971	5.0000	4.4808	201	11/02/2013	0.0019	0.0223	199	6836	205	4.4298
5	12/02/2013	0.8513	8097	5.0000	4.4729	200	12/02/2013	0.0018	0.0223	200	6776	203	4.4727
6	13/02/2013	0.8294	8201	5.0000	4.4621	200	13/02/2013	0.0017	0.0223	197	6792	204	4.4616
7	14/02/2013	0.9903	3812	5.0000	4.4573	199	14/02/2013	0.0017	0.0223	114	3812	114	2.5463
Total			52846	35.0000							44544	1335	33.0362

on the corresponding delivery dates. We use the actual bids from RTB to simulate the real-time feeds of the spot market, so if the market value of a click is higher than the fixed payment, the advertiser will use ad options to secure the needed clicks and then pay the fixed CPCs accordingly. Otherwise, the advertiser will obtain the equivalent clicks from RTB. Our example shows a "bull market" where the average spot CPM in the test set is far higher than the initial CPM. Therefore, ad options would be actively used by the advertiser to purchase the clicks. Compared to Table 8, the advertiser can receive more clicks (increased by 20.92%) in a bull market via ad options.

Similar experiments are conducted for all ad slots in our datasets. The overall results are presented in Tables 10 and 11. For the SSP dataset, we consider the ad options that allow advertisers to pay a fixed CPC to purchase impressions of targeted ad slots. For the Google dataset, we consider the ad options that allow advertisers to pay a fixed CPM to purchase clicks of their targeted keywords. To summarize, we find that an advertiser's daily budget can be used more effectively in a bull market and that his delivery increases as well. The advertiser's average cost spent on each impression or click is reduced. In a bear market (i.e., the underlying price decreases), the advertiser will use the ad options less (and sometimes not at all) and the maximum cost is just the option price. It is worth noting that here we consider the ad options are in the money at time 0 (i.e., the strike price is less than the current underlying price). In Table 8, there are 4 ad slots that exhibit somewhat bear markets. However, these 4 ad slots do not receive enough bids in the test set and the actual winning payment CPMs are just around its floor reserve level (i.e., the CPM is £0.01 so the per impression price is £0.00001). Since these prices will seriously bias the results, we do not take them into account in the situation of a bear market.

Table 10
Overview of the improvement in delivery performance by using ad options for all ad slots in the SSP dataset.

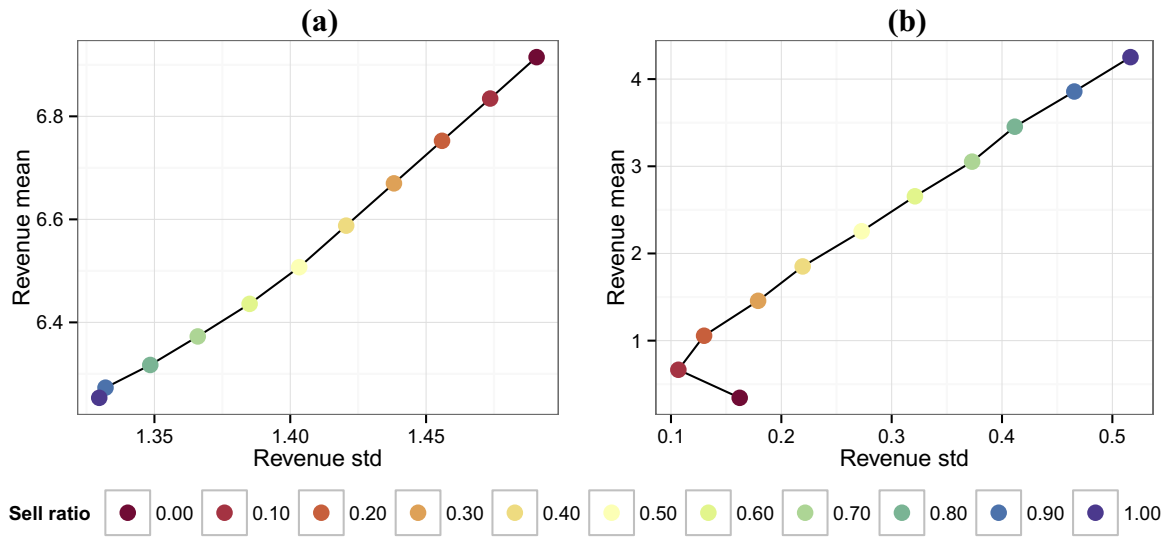|	Bull market	Bear market
Change on used budget (%)	−8.7878	–
Change on delivery of impressions (%)	6.1781	–

Table 11
Overview of the improvement in delivery performance by using ad options for keywords in the Google AdWords dataset.

Market	Group	Change in used budget (%)		Change in delivery of impressions (%)	
		Bull market	Bear market	Bull market	Bear market
US	1	0.3447	2.3438	9.3050	−0.1122
	2	1.7748	3.9687	2.3153	−2.6285
	3	0.5372	4.8567	44.3735	−0.0940
	4	5.6288	29.3626	1.6433	−1.0993
UK	1	21.4285	6.8940	3.0717	−0.2523
	2	5.4426	0.0000	0.4419	0.0000
	3	10.9285	3.8474	28.7706	−2.1066
	4	6.7155	0.1552	16.6955	−2.1550

5.5. Revenue analysis for publisher and search engine

We also investigate the revenue effects when a certain amount of future impressions or clicks is sold in advance. Fig. 11 provides two empirical examples of ad slots from the SSP dataset: one exhibits the bull market while the other shows the bear market. The sell ratio in the figure represents the percentage of future impressions

Fig. 11. Empirical examples of the publisher's revenue: (a) from an ad slot in the bull market; and (b) from an ad slot in the bear market. The sell ratio represents the percentage of future daily impressions that are sold in advance via display ad options. Note that here the ad slot in the bear market does not receive enough bids in the test set, so we randomly simulate some underlying prices for the bear market.

that are sold in advance via ad options; therefore, when the sell ratio equals zero, the publisher auctions off all of the future impressions in RTB. Fig. 11(a) suggests that the publisher should sell less future impressions in advance if the future market is bull. This is because ad options will be exercised by advertisers in the future and the obtained revenues from the fixed payment are less than these impressions' market values. Of course, the publisher can choose a certain percentage of future impressions to sell according to his level of risk tolerance or to meet other business objectives. For example, the publisher may be willing to sacrifice some revenues in order to increase the advertisers' engagement in the long run. Conversely, in a bear market, as shown in Fig. 11(b), the publisher is advised to sell more future impressions in advance because there is more upfront income if more display ad options are sold, and in the future advertisers will not exercise the sold options. Therefore, the increased revenue comes from the option price.

Based on the above analysis, we examine the revenue effects across all ad slots and keywords in our datasets. In the experiments, the display ad options in a bull market are priced in the money while in a bear market they are priced out of the money. The sell ratio is set at 0.20 in a bull market while it is set at 0.80 in a bear market. The overall results are presented in Tables 12 and 13, which further confirm our analysis in the empirical examples. The average revenue is reduced in the bull market as well as the standard deviation (i.e., one kind of revenue risk). However, as described, the publisher (or search engine) may be willing to sacrifice some revenue to establish a long-term relationship with advertisers. In a bear market, the average revenue increases significantly. This is because fewer display ad options are exercised. Many premium advertisers join RTB so that the market equilibrium is almost as same as that in an environment with only auctions. Finally, the publisher (or search engine) earns the upfront payment without providing guaranteed deliveries.

Table 12

Overview of the improvement in revenue by selling display ad options for ad slots in the SSP dataset.

	Bull market	Bear market
Change on mean (%)	−7.1283	726.3085
Change on standard deviation (%)	−2.7041	196.0547

Table 13

Overview of the improvement in revenue by selling display ad options for ad slots in the Google AdWords dataset.

Market	Group	Change in mean (%)		Change in standard deviation (%)	
		Bull market	Bear market	Bull market	Bear market
US	1	−20.5880	22.3898	−0.6507	9.3291
	2	−23.2971	17.1898	−17.6508	9.4175
	3	−32.8388	69.9113	−21.9468	−2.1065
	4	−24.4710	8.9650	−10.6024	95.4868
UK	1	−8.5463	15.4155	4.5617	10.4116
	2	−20.0632	4.3816	−16.0239	6.8847
	3	−16.9050	30.7737	−11.4811	−19.4625
	4	−21.8142	7.6342	−19.4368	0.3877

6. Concluding remarks

This paper described a new ad option tailored to the unique environment of display advertising. A binomial lattice framework with censored probabilities was proposed to price the ad option where the underlying prices follow a SV model. We also reviewed and examined several lattice methods for pricing the ad option with the GBM underlying model. Our developments were examined and validated by experiments using real advertising data.

We believe that the proposed ad options will soon be welcomed by display advertising market. Several similar but different developments appeared are able to support our point of view. They are:

09/2013 AOL's Programmatic Upfront.[2]
03/2013 OpenX Programmatic Guarantee.[3]
10/2012 Adslot Media's Programmatic Direct Media Buying.[4]
10/2012 Shiny Ads Direct's End-to-end Programmatic Direct Advertising Platform.[5]
10/2012 iSOCKET's Programmatic Direct.[6]

---

[2] www.aolplatforms.com.
[3] www.openx.com.
[4] www.automatedguaranteed.com.
[5] www.shinyads.com.
[6] www.isocket.com.

Our work differs to the above developments in many aspects. First, the proposed ad options provide flexible guaranteed deliveries (e.g., no obligation of exercise, choosing the fixed payment that is different to the underlying inventory measurement model) while other recent developments do not provide such features. Second, we proposed a generalized pricing model which can deal with those situations when the GBM model fails.

There are three major limitations of the study in this paper, which can be further explored for future research. Firstly, we did not explicitly consider the capacity issue in option pricing. Therefore, there may exist the situations that a publisher or search engine can not guarantee the delivery of impressions or clicks sold by options. In our current study, we consider the seller has a good estimation of the inventories that will be created in the future and rationally sells the future inventories in advance via options. If the seller over sells the future inventories, we also assume that he can buy some similar inventories on the spot market once the option buyers request to exercise the options. In such case, the revenues of the seller will decrease. The capacity issue is an interesting topic to further discuss in details because it has two challenges. The first challenge is to price an ad option with explicitly considering the estimation of future supply and demand of inventories, where the latter two variables can be described to be static (Wang and Chen 2012) or dynamic like the Poisson process (Gallego and van Ryzin 1994). The second challenge is considering the penalty into option pricing. If the seller fails to deliver inventories requested by option holders, the seller should pay a certain amount of penalty fee (Chen et al. 2014). However, with the penalty setting, some advertisers who only pursue the penalty may game the system (Constantin et al. 2009), which will further affect the calculated option price, and such effect will also generate some scenarios like the implied volatility in financial market. The second limitation is that the proposed model can not capture the jumps and volatility clusters of underlying inventory prices. It might be of interest to discuss these stylized facts in ad option pricing. The third limitation is the zero correlation of the two standard Brownian motions in our proposed dynamics. If their correlation is not zero, the option pricing would be more sophisticated under the lattice framework. Heston (1993) proposed a good solution in the continuous-time settings, which can also be extended to solve our problem in online advertising.

References

Bachelier, L., 1900. Théorie de la spéculation. Annales Scientifiques de l'École Normale Supérieure 3 (17), 21–86.

Bharadwaj, V., Ma, W., Schwarz, M., Shanmugasundaram, J., Vee, E., Xie, J., Yang, J., 2010. Pricing guaranteed contracts in online display advertising. In: Proceedings of the 19th ACM International Conference on Information and Knowledge Management. ACM, Toronto, ON, Canada, pp. 399–408.

Björk, T., 2009. Arbitrage Theory in Continuous Time, 3rd edition. Oxford University Press.

Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81 (3), 637–654.

Boer, P., 2002. The Real Options Solution Finding Total Value in a High-Risk World. John Wiley.

Boyle, P., 1986. Option valuation using a three-jump process. International Options Journal 3, 7–12.

Boyle, P., 1988. A lattice framework for option pricing with two state variables. Journal of Financial and Quantitative Analysis 23 (1), 1–12.

Chen, B., Wang, J., Cox, I., Kankanhalli, M., 2015. Multi-keyword multi-click advertisement option contracts for sponsored search. ACM Transactions on Intelligent Systems and Technology. http://dx.doi.org/10.1145/2743027.

Chen, B., Yuan, S., Wang, J., 2014. A dynamic pricing model for unifying programmatic guarantee and real-time bidding in display advertising. In: Proceedings of the 8th International Workshop on Data Mining for Online Advertising. ACM, New York, NY, USA, pp. 1–9.

Constantin, F., Feldman, J., Muthukrishnan, S., Pál, M., 2009. An online mechanism for ad slot reservations with cancellations. In: Proceedings of the 20th Annual ACM-SIAM Symposium on Discrete Algorithms. SIAM, New York Marriott Downtown, NY, USA, pp. 1265–1274.

Cox, J., Ross, S., Rubinstein, M., 1979. Option pricing: a simplified approach. Journal of Financial Economics 7, 229–263.

Edelman, B., Ostrovsky, M., Schwarz, M., 2007. Internet advertising and the generalized second-price auction: selling billions of dollars worth of keywords. American Economic Review 97 (1), 242–259.

Florescu, I., Viens, F., 2005. A binomial tree approach to stochastic volatility driven model of the stock price. Annals of University of Craiova 32, 126–142.

Florescu, I., Viens, F., 2008. Stochastic volatility: option pricing using a multinomial recombining tree. Applied Mathematical Finance 15 (2), 151–181.

Gallego, G., van Ryzin, G., 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. Management Science 40 (8), 999–1020.

Glasserman, P., 2003. Monte Carlo Methods in Financial Engineering. Springer.

Google, 2011. The arrival of real-time bidding and what it means for media buyers. White Paper.

Haahtela, T. Recombining trinomial tree for real option valuation with changing volatility. doi: http://dx.doi.org/10.2139/ssrn.1932411, 2010.

Heston, S., 1993. A closed-form solution for. options with stochastic volatility with applications to bond and currency options. The Review of Financial Studies 6 (2), 327–343.

Interactive Advertising Bureau of Canada. Real-time bidding glossary. http://iabcanada.com/files/RTB-Glossary-English.pdf, 2015.

Kamrad, B., Ritchken, P., 1991. Multinomial approximating models for options with k state variables. Management Science 37 (23), 1640–1652.

Ljung, G., Box, G., 1978. On a measure of lack of fit in time series models. Biometrika 65 (2), 297–303.

Marathe, R., Ryan, S., 2005. On the validity of the geometric Brownian motion assumption. The Engineering Economist 50, 159–192.

Marshall, J. Valuation of multiple exercise options. Ph.D. thesis, University of Western Ontario, 2012.

Merton, R., 1973. The theory of rational option pricing. Bell Journal of Economics and Management Science 4 (1), 141–183.

Moon, Y., Kwon, C., 2010. Online advertisement service pricing and an option contract. Electronic Commerce Research and Applications 10 (1), 38–48.

Nelson, D., Ramaswamy, K., 1990. Simple binomial processes as diffusion aapproximation in financial models. The Review of Finanical Studies.

Primbsa, J., Rathinamb, M., Yamadac, Y., 2007. Option pricing with a pentanomial lattice model that incorporates skewness and kurtosis. Applied Mathematical Finance, 1–17.

Samuelson, P., 1965. Rational theory of warrant pricing. Industrial Management Review 6, 13–31.

Shapiro, S., Wilk, M., 1965. An analysis of variance test for normality. Biometrika 52 (3), 591–611.

Sharpe, W., 1978. Investments, 1st edition. Englewood Cliffs.

Tian, Y., 1993. A modified lattice approach to option pricing. The Journal of Futures Markets.

Tsay, R., 2005. Analysis of Financial Series, 2nd edition. John Wiley.

Varian, H., 1987. The arbitrage principle in financial economics. Journal of Economic Perspectives 1 (2), 55–72.

Wang, J., Chen, B., 2012. Selling futures online advertising slots via option contracts. In: Proceedings of the 21st International Conference on World Wide Web. ACM, Lyon, France, pp. 627–628.

Wilmott, P., 2006. Paul Wilmott On Quantitative Finance, 2nd edition. John Wiley.

Yuan, S., Wang, J., 2012. Sequential selection of correlated ads by POMDPs. In: Proceedings of the 21st ACM International Conference on Information and Knowledge Management. ACM, Maui, HI, USA, pp. 515–524.

Yuan, S., Wang, J., Chen, B., Mason, P., Seljan, S., 2014. An empirical study of reserve price optimisation in real-time bidding. In: Proceedings of the 20th ACM SIGKDD Conference on Knowledge Discovery and Data Mining. ACM, New York, NY, USA, pp. 1897–1906.

Yuan, S., Wang, J., Zhao, X., 2013. Real-time bidding for online advertising: measurement and analysis. In: Proceedings of the 7th Workshop on Data Mining for Online Advertising. ACM, Chicago, IL, USA, pp. 1897–1906.

Zhang, P., 1998. Exotic Options, 2nd edition. World Scientific.