A lattice framework for pricing display advertisement options with the stochastic volatility underlying model

Supplementary material

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This supplementary material is to provide some mathematical preliminaries and details for the discussion in Section 3 and Eq.(13) in Section 4.

1 Equivalence of the option prices from a risk-neutral advertiser and a risk-averse publisher under the one-step binomial lattice

Under the one-step binomial lattice, we now derive the option pricing formula from the perspective of a publisher who wants to hedge the revenue risk incurred from price changes. The derived option price is equal to the one that is calculated from the perspective of a risk-neutral advertiser in the paper.

Consider the case where an ad option allows its buyer to pay a fixed CPC for display impressions. Therefore, the strike price of the option is the fixed CPC and the underlying price is the uncertain winning payment CPM in online auctions. Suppose that there is a deterministic total number of future impressions to sell, denoted by S^M. If the CPM in time 1 goes up, the publisher's revenue can be expressed as

$$R_1^{\{u\}} = \begin{cases} (1-\alpha)S^M/1000M_1^{\{u\}} + \alpha S^M H F^M, & \text{if } M_1^{\{u\}} \geq F^C, \\ (1-\alpha)S^M/1000M_1^{\{u\}} + \alpha S^M/1000M_1^{\{u\}}, & \text{if } M_1^{\{u\}} < F^C, \end{cases} \tag{1}$$

where α is the percentage of estimated total impressions to sell via ad options. Eq. (1) shows that the publisher's revenue is a combination of guaranteed and non-guaranteed impressions. Eq. (1) can be rewritten as $R_1^{\{u\}} = S^M/1000M_1^{\{u\}} - \alpha S^M H \Phi_1^{\{u\}}$, where $\Phi_1^{\{u\}}$ is the option payoff function, defined by $\max\{M_1^{\{u\}}/(1000H) - F^C, 0\}$, and the superscript notation $\{u\}$ represents the upward movement. Similarly, if CPM in time 1 goes down, the publisher's revenue is $R_1^{\{d\}} = S^M/1000M_1^{\{d\}} - \alpha S^M H \Phi_1^{\{d\}}$, where $\Phi_1^{\{d\}} = \max\{M_1^{\{d\}}/(1000H) - F^C, 0\}$.

Since the publisher uses α to control the revenue in bull and bear markets, there exists a value α^* such that $R_1^{\{u\}}(\alpha^*) = R_1^{\{d\}}(\alpha^*)$, then $\alpha^* = (M_1^{\{u\}} - M_1^{\{d\}})/(\Phi_1^{\{u\}} - \Phi_1^{\{d\}})$. As described, the publisher's least requirement on the valuation is that his expected future revenue (including the upfront income in terms of option prices) should be equal to the current revenue level from auctions alone, so the following equation holds:

$$R_0 = \frac{\alpha^* S^M}{1000}\pi_0 + \widetilde{r}^{-1} R_1^{\{u\}}(\alpha^*) = \frac{\alpha^* S^M}{1000}\pi_0 + \widetilde{r}^{-1} R_1^{\{d\}}(\alpha^*).$$

The option price π_0 can then be calculated by

$$\pi_0 = \widetilde{r}^{-1}\left(\frac{\widetilde{r}M_0 - M_1^{\{d\}}}{M_1^{\{u\}} - M_1^{\{d\}}}\Phi_1^{\{u\}} + \frac{M_1^{\{u\}} - \widetilde{r}M_0}{M_1^{\{u\}} - M_1^{\{d\}}}\Phi_1^{\{d\}}\right)$$
$$= \widetilde{r}^{-1}\left(\frac{\widetilde{r} - d}{u - d}\Phi_1^{\{u\}} + \frac{u - \widetilde{r}}{u - d}\Phi_1^{\{d\}}\right),$$

where $u = M_1^{\{u\}}/M_0, d = M_1^{\{u\}}/M_0$. Up to this point, we have proved that the calculated option price π_0 is no-arbitrage and hedges the revenue for the publisher.

2 Convergence of the Binomial lattice option pricing model to the Black-Scholes-Merton model

Consider the case where an ad option allows its buyer to pay a fixed CPC for display impressions. Hence, the strike price of the option is the fixed CPC and the underlying price is the uncertain winning payment CPM in online auctions. Let $\Delta t = T/n, u = e^{\sigma\sqrt{\Delta t}} > 1, d = 1/u < 1$, where σ is the volatility of CPM. Let r be the constant continuous-time risk-less interest rate and let $M(t)$ be the continuous-time CPM at time t. Under the risk-neutral probability measure $\mathbb{Q}$, the GBM underlying can be expressed as

$$dM(t) = rM(t)dt + \sigma M(t)dW^{\mathbb{Q}}(t),$$

where $W^{\mathbb{Q}}(t)$ is a standard Brownian motion under $\mathbb{Q}$.

As described, for $j \geq j^*, j = 1, 2, \ldots, n, j^* = 1, 2, \ldots, n$, the advertiser will exercise the option to buy the targeted impressions, so we have the following inequality:

$$\frac{M_0}{1000H}u^{j^*-1}d^{n-j^*+1} < F^C \leq \frac{M_0}{1000H}u^{j^*}d^{n-j^*},$$

then

$$\left(\frac{u}{d}\right)^{j^*-1} \leq \frac{1000HF^C}{M_0 d^n} \leq \left(\frac{u}{d}\right)^{j^*},$$

and, taking logarithms and dividing by $\ln(u/d)$ and subtracting nq from each side gives

$$\frac{j^* - 1 - nq}{\sqrt{n}} \leq \frac{\ln(1000HF^C/M_0) - \ln(u^{nq}d^{n(1-q)})}{\sqrt{n}(\ln(u) - \ln(d))} \leq \frac{j^* - nq}{\sqrt{n}}.$$

Since $\lim_{n\to\infty} \frac{j^*-nq}{\sqrt{n}} - \frac{j^*-1-nq}{\sqrt{n}} = 0$, then

$$\frac{\ln(1000HF^C/M_0) - \ln(u^{nq}d^{n(1-q)})}{\sqrt{n}(\ln(u) - \ln(d))} \approx \frac{j^* - nq}{\sqrt{n}}.$$

Hence, we can obtain

$$\psi(j^*, n, q) = \mathbb{P}(j \geq j^*, j \in \{1, \ldots, n\}) = \mathbb{P}\left[\frac{j - nq}{\sqrt{nq(1-q)}} \geq \frac{j^* - nq}{\sqrt{nq(1-q)}}\right]$$
$$=1 - \mathcal{N}\left[\frac{j^* - nq}{\sqrt{nq(1-q)}}\right] = \mathcal{N}\left[\frac{nq - j^*}{\sqrt{nq(1-q)}}\right]$$
$$= \mathcal{N}\left[\frac{\ln(M_0/(1000HF^c)) + \ln(u^{nq}d^{n(1-q)})}{\sqrt{n}(\ln(u) - \ln(d))\sqrt{q(1-q)}}\right],$$

where $\mathcal{N}[\cdot]$ is the cumulative distribution function of a standard normal distribution.

If $n \to \infty$ (or $\Delta t \to 0$), the following convergence results can be obtained:

$$\begin{aligned}
\sqrt{n}\ln(u) &= \lim_{n\to\infty} \sqrt{n}\ln e^{\sigma\sqrt{\Delta t}} = \sigma\sqrt{T}, \\
\sqrt{n}\ln(d) &= \lim_{n\to\infty} \sqrt{n}\ln e^{-\sigma\sqrt{\Delta t}} = -\sigma\sqrt{T}, \\
q = \frac{\widetilde{r}-d}{u-d} &= \lim_{\Delta t\to 0} \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} = \lim_{\Delta t\to 0} \frac{\sigma + (r - \frac{1}{2}\sigma^2)\sqrt{\Delta t} + o(\Delta^{3/2})}{2\sigma + o(\Delta t)} = \frac{1}{2}, \\
\lim_{\Delta t\to 0} \frac{2q-1}{\sqrt{\Delta t}} &= \lim_{\Delta t\to 0} \frac{2}{\sqrt{\Delta t}} \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} = \frac{r}{\sigma} - \frac{\sigma}{2},
\end{aligned}$$

and then

$$\begin{aligned}
\ln(u^{nq}d^{n(1-q)}) &= nq\ln(u) + n(1-q)\ln(d) = nq\sigma\sqrt{\Delta t} - n(1-q)\sigma\sqrt{\Delta t} = 2nq\sigma\sqrt{\Delta t} - n\sigma\sqrt{\Delta t} \\
&\approx 2n\sigma\sqrt{\Delta t} \lim_{\Delta t\to 0} \frac{\sigma + (r - \frac{1}{2}\sigma^2)\sqrt{\Delta t} + o(\Delta t)}{2\sigma + o(\Delta t)} - \lim_{\Delta\to 0} n\sigma\sqrt{\Delta t} \\
&\approx \lim_{\Delta t\to 0} n\sigma\sqrt{\Delta t} + (r - \frac{1}{2}\sigma^2)T - \lim_{\Delta t\to 0} n\sigma\sqrt{\Delta t} = (r - \frac{1}{2}\sigma^2)T.
\end{aligned}$$

Therefore,

$$\psi(j^*, n, q) = \mathcal{N}\left[\frac{\ln(M_0/(1000HF^C)) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right].$$

As described $\widetilde{q} = qu/\widetilde{r}$, then

$$\begin{aligned}
&\ln(u^{n\widetilde{q}}d^{n(1-\widetilde{q})}) - \ln(u^{nq}d^{n(1-q)}) \\
&= \ln(u^{n(\widetilde{q}-q)}d^{n(q-\widetilde{q})}) = n(\widetilde{q}-q)\ln(u) + n(q-\widetilde{q})\ln(d) = n(\widetilde{q}-q)\sigma\sqrt{\Delta t} - n(q-\widetilde{q})\sigma\sqrt{\Delta t} \\
&= 2n(\widetilde{q}-q)\sigma\sqrt{\Delta t} = 2nq\frac{u-\widetilde{r}}{\widetilde{r}}\sigma\sqrt{\Delta t} = 2q\sigma\frac{T}{\sqrt{\Delta t}}\frac{e^{\sigma\sqrt{\Delta t}} - e^{r\Delta t}}{e^{r\Delta t}} \approx \lim_{\Delta t\to 0} 2q\sigma T\frac{\sigma - r\Delta t}{1 + r\Delta t} = \sigma^2 T.
\end{aligned}$$

Similarly, $\widetilde{q} = \lim_{\Delta t\to 0} \frac{1+\sigma\sqrt{\Delta}}{1+\widetilde{\Delta} t}\frac{\sigma + (r-\frac{1}{2}\sigma^2)\sqrt{\Delta t})}{2\sigma} = \frac{1}{2}$, then

$$\begin{aligned}
\psi(j^*, n, \widetilde{q}) &= \mathcal{N}\left[\frac{\ln(M_0/(1000HF^C)) + \ln(u^{n\widetilde{q}}d^{n(1-\widetilde{q})})}{\sqrt{n}(\ln(u) - \ln(d))\sqrt{\widetilde{q}(1-\widetilde{q})}}\right] \\
&= \mathcal{N}\left[\frac{\ln(M_0/(1000HF^C)) + \ln(u^{nq}d^{n(1-q)}) + \left(\ln(u^{n\widetilde{q}}d^{n(1-\widetilde{q})}) - \ln(u^{nq}d^{n(1-q)})\right)}{\sqrt{n}(\ln(u) - \ln(d))\sqrt{\widetilde{q}(1-\widetilde{q})}}\right] \\
&= \mathcal{N}\left[\frac{\ln(M_0/(1000HF^C)) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right].
\end{aligned}$$

The continuous-time option pricing formula can be then obtained:

$$\begin{aligned}
\pi_0 &= \frac{M_0}{1000H}\mathcal{N}[\varsigma_1] - F^C e^{-rT}\mathcal{N}[\varsigma_2], \\
\varsigma_1 &= \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{M_0}{1000HF^C}\right) + (r + \frac{1}{2}\sigma^2)T\right), \\
\varsigma_2 &= \varsigma_1 - \sigma\sqrt{T}.
\end{aligned}$$

Hence, if the GBM assumption is valid empirically, one can use the closed-form solution to calculate the option price. However, as described, lattice methods provide an alternative way to calculate the option price and, in general, are simpler in terms of calculation.

3 Pricing an ad option with the geometric Brownian motion underlying using a trinomial lattice

Below the algorithm simply shows how to calculate the option price of a display ad option with the GBM underlying over a trinomial lattice. Here we consider the case where an ad option allows its buyer to pay the fixed CPC for display impressions. Hence, the strike price of the option is the fixed CPC and the underlying price is the uncertain winning payment CPM from online auctions. One can slightly modify the algorithm and calculate the option price over a binomial lattice.

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function OPTIONPRICINGTRINOMIALLATTICE($M_0, \sigma, H, T, n, r, F^C$)
 # Initialization:
 $\Delta t \leftarrow T/n$; $\widetilde{r} \leftarrow e^{r\Delta t}$;
 $u, m, d, q_1, q_2, q_3 \leftarrow$ Boyle-TRIN (or KR-TRIN or Tian-TRIN) in Table 1;
 # Build a (recombining) trinomial lattice for CPM
 $\mathbf{\Sigma}_{(n+1)\times(n+1)} \leftarrow \mathbf{0}_{(n+1)\times(n+1)}$; $\mathbf{\Sigma}_{(1,1)} \leftarrow M_0$;
 for $j \leftarrow 2$ to $n+1$ **do**
  $\mathbf{\Sigma}_{(1,j)} \leftarrow u \times \Sigma_{(1,j-1)}$; $\mathbf{\Sigma}_{(2,j)} \leftarrow m \times \Sigma_{(1,j-1)}$; $\mathbf{\Sigma}_{(3,j)} \leftarrow d \times \Sigma_{(1,j-1)}$;
  if $2(j-1)+1 > 3$ **then**
   for $k \leftarrow 4$ to $2(j-1)+1$ **do**
    $\mathbf{\Sigma}_{(k,j)} \leftarrow d \times \mathbf{\Sigma}_{(k-2,j-1)}$;
   end for
  end if
 end for
 # Calculate the terminal payoffs and the option value backward recursively
 $\widetilde{\mathbf{\Sigma}}_{(n+1)\times(n+1)} \leftarrow \mathbf{0}_{(n+1)\times(n+1)}$; $\widetilde{\mathbf{\Sigma}}_{(:,n+1)} \leftarrow \max\{\mathbf{\Sigma}_{(:,n+1)}/(1000H) - F^C, 0\}$;
 for $j \leftarrow n$ to 1 **do**
  for $k \leftarrow 1$ to $2(j-1)+1$ **do**
   if $k = 1$ **then**
    $\widetilde{\mathbf{\Sigma}}_{(k,j)} \leftarrow \widetilde{r}^{-1}(q_1\widetilde{\mathbf{\Sigma}}_{(k,j+1)} + q_2\widetilde{\mathbf{\Sigma}}_{(k,j+1)} + q_3\widetilde{\mathbf{\Sigma}}_{(k,j+1)})$;
   else if $k \geq 2$ **then**
    $\widetilde{\mathbf{\Sigma}}_{(k,j)} \leftarrow \widetilde{r}^{-1}(q_1\widetilde{\mathbf{\Sigma}}_{(k-1,j+1)} + q_2\widetilde{\mathbf{\Sigma}}_{(k,j+1)} + q_3\widetilde{\mathbf{\Sigma}}_{(k+1,j+1)})$;
   end if
  end for
 end for
 return $\pi_0 \leftarrow \widetilde{\mathbf{\Sigma}}_{(1,1)}$
end function

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4 Risk-neutral probability measure for the SV model

For the stochastic volatility model:

$$dM(t) = \mu M(t)dt + \sigma(t)M(t)dW(t), \tag{2}$$

$$d\sigma(t) = \kappa(\theta - \sigma(t))dt + \delta\sqrt{\sigma(t)}dZ(t), \tag{3}$$

where $W(t)$ and $Z(t)$ are standard Brownian motions under the real world probability measure $\mathbb{P}$ satisfying $\mathbb{E}[dW(t)dZ(t)] = 0$, and μ and $\sigma(t)$ are the constant drift and volatility of CPM, and κ, θ, δ are the volatility parameters. The drift factor $\kappa(\theta - \sigma(t))$ ensures the mean reversion of $\sigma(t)$ towards its long-term value θ. The volatility factor $\delta\sqrt{\sigma(t)}$ avoids the possibility of negative $\sigma(t)$ for all positive values of κ and θ.

By applying the Itô Lemma to Eq. (2), we obtain

$$M(t) = M(0)\exp\left\{\int_0^t (\mu - \frac{1}{2}\sigma^2(t))ds + \int_0^t \sigma(s)dW(s)\right\}.$$

Consider a discount process $D(t) = e^{-rt}$, where r is a constant risk-less interest rate. Then $dD(t) = -rD(t)dt$. Therefore, the discounted CPM process is

$$D(t)M(t) = M(0)\exp\left\{\int_0^t (\mu - rD(t) - \frac{1}{2}\sigma^2(t))ds + \int_0^t \sigma(s)dW(s)\right\},$$

and its differential is

$$\begin{aligned} d\Big(D(t)M(t)\Big) &= M(t)dD(t) + D(t)dM(t) \\ &= -rD(t)M(t)dt + D(t)(\mu M(t)dt + \sigma(t)M(t)dW(t)) \\ &= \sigma(t)D(t)M(t)\left(\frac{\mu - r}{\sigma(t)}dt + dW(t)\right) \\ &= \sigma(t)D(t)M(t)dW^{\mathbb{Q}}(t), \end{aligned}$$

where $W^{\mathbb{Q}}(t) = W(t) + \int_0^t \frac{\mu - r}{\sigma(s)}ds$. According to the Girsanov Theorem, if choosing the process $\tau(t) = \frac{\mu - r}{\sigma(t)}$, then $W^{\mathbb{Q}}(t)$ is a standard Brownian motion under a new probability measure $\mathbb{Q}$. We know that $\mathbb{Q}$ is risk-neutral because it renders $D(t)M(t)$ into a martingale. The risk-neutral formulation of Eq. (2) is then

$$dM(t) = rM(t)dt + \sigma(t)M(t)dW^{\mathbb{Q}}(t).$$

Hence $dX(t) = (r - \sigma^2(t)/2)dt + \sigma(t)dW^{\mathbb{Q}}(t)$ if $X(t) = \ln\{M(t)\}$ (see Eq. (13) in Section 4).