Multi-Keyword Multi-Click Advertisement Option Contracts for Sponsored Search

BOWEI CHEN and JUN WANG, University College London
INGEMAR J. COX, University of Copenhagen and University College London
MOHAN S. KANKANHALLI, National University of Singapore

In sponsored search, advertisement (abbreviated ad) slots are usually sold by a search engine to an advertiser through an auction mechanism in which advertisers bid on keywords. In theory, auction mechanisms have many desirable economic properties. However, keyword auctions have a number of limitations including: the uncertainty in payment prices for advertisers; the volatility in the search engine's revenue; and the weak loyalty between advertiser and search engine. In this article, we propose a special ad option that alleviates these problems. In our proposal, an advertiser can purchase an option from a search engine in advance by paying an upfront fee, known as the option price. The advertiser then has the right, but no obligation, to purchase among the prespecified set of keywords at the fixed cost-per-clicks (CPCs) for a specified number of clicks in a specified period of time. The proposed option is closely related to a special exotic option in finance that contains multiple underlying assets (multi-keyword) and is also multi-exercisable (multi-click). This novel structure has many benefits: advertisers can have reduced uncertainty in advertising; the search engine can improve the advertisers' loyalty as well as obtain a stable and increased expected revenue over time. Since the proposed ad option can be implemented in conjunction with the existing keyword auctions, the option price and corresponding fixed CPCs must be set such that there is no arbitrage between the two markets. Option pricing methods are discussed and our experimental results validate the development. Compared to keyword auctions, a search engine can have an increased expected revenue by selling an ad option.

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1. INTRODUCTION

Sponsored search has become an important online advertising format [PWC 2013], in which a search engine sells ad slots in the search engine results pages (SERPs)

generated in response to a user's search behaviour. An online user submits a term or phrase within the search box to the search engine. The term or phrase is collectively known as the *query*. The SERP has two types of result listings in response to the submitted query: organic results and paid results. *Organic results* are the Web page listings that most closely match the user's search query based on relevance [Jansen 2011]. *Paid results* are online ads – the companies who have paid to have their Web pages displayed for certain keywords, so that such listings show up when a user submits a search query containing those keywords. The price of an ad slot is usually determined by a keyword auction such as the widely used *generalized second price* (GSP) auction [Edelman et al. 2007; Varian 2007; Lahaie and Pennock 2007; Börgers et al. 2013; Qin et al. 2014]. In the GSP auction, advertisers bid on keywords present in the query, and the highest bidder pays the price associated with the second highest bid.

Despite the success of keyword auctions, there are two major drawbacks. First, the uncertainty and volatility of bids make it difficult for advertisers to predict their campaign costs and thus complicate their business planning [Wang and Chen 2012]. Second, the "pay-as-you-go" nature of auction mechanisms does not encourage a stable relationship between advertiser and search engine [Jank and Yahav 2010] – an advertiser can switch from one search engine to another in the next bidding at near-zero cost.

To solve these problems, we propose a *multi-keyword multi-click ad option*. It is essentially a contract between an advertiser and a search engine. It consists of a nonrefundable upfront fee, known as the *option price*, paid by the advertiser, in return for the right, but not the obligation, to subsequently purchase a fixed number of clicks for particular keywords for a prespecified fixed cost-per-clicks (CPCs) during a specified period of time. From the advertiser's perspective, fixing the CPCs significantly reduces the uncertainty in the cost of advertising campaigns. Moreover, for a keyword, if the spot CPC set by keyword auction falls below the fixed CPC of the option contract, the advertiser is not obligated to exercise the option, but can, instead, participate in keyword auctions. Therefore, the option can be considered as an "insurance" that establishes an upper limit on the cost of advertising campaigns. From the search engine's perspective, the proposed option is not only an additional service provided for advertisers. We show that the search engine can, in fact, increase the expected revenue in the process of selling an ad option. Also, since the option covers a specific period of time, this should encourage a more stable relationship between advertiser and search engine.

An important question for us is to determine the option price and the fixed CPCs associated with candidate keywords in the advertiser's request list. Clearly, if the option is priced too low, then significant loss in revenue to the search engine may result. Moreover, this may create an arbitrage opportunity in which the buyer of the option sells the clicks from one's targeted keywords to gain extra profits. Conversely, if the option is priced too high, then the advertiser will not purchase it. In this article, we consider a risk-neutral environment and price the option under the no-arbitrage objective [Wilmott 2006; Björk 2009]. We use the Monte Carlo method to price the option with multiple candidate keywords and show the closed-form pricing formulas for the cases of single keyword and two keywords. Further, the effects of ad options on the search engine's revenue is analysed.

This article has three major contributions. First, we propose a new way to presell ad slots in sponsored search, which provides flexible guaranteed deliveries to advertisers. It naturally complements the current keyword auction mechanism and offers both advertiser and search engine an effective risk mitigation tool to deal with fluctuations in the bid price. Although the proposed ad option belongs to a family of exotic options, it differs from existing exotic options in that we know the following from finance and

other industries (see Table I for detailed comparisons): it can be exercised not only once but multiple times during the contract period; it is not for a single keyword but multiple keywords, and each keyword has its own fixed CPC; it allows its buyer to choose which keyword to reserve and advertise at the corresponding fixed CPC later during the contract period. Second, we discuss a generalized pricing method for the proposed ad option (see Algorithm 1) to deal with the high dimensionality. Third, we demonstrate that, compared to keyword auctions, a search engine can have an increased expected revenue by selling an ad option.

The rest of the article is organised as follows. Section 2 reviews the related literature. Section 3 introduces the design of proposed ad option, discusses the option pricing methods, and analyses the option effects on the search engine's revenue. Section 4 presents our empirical evaluation, and Section 5 contains our conclusions. Several important mathematical results are provided in Appendices A through C.

2. RELATED WORK

The work presented in this article touches upon several streams of literature. We first review the prior work on options in finance and other industries, then discuss the related literature in guaranteed advertising deliveries.

2.1. Options and Their Pricing Methods

Options have been known and traded for many centuries, and can be traced back to the 17th century [Constantinides and Malliaris 2001]. A *standard option* is a contract in which the seller grants the buyer the right, but not the obligation, to enter into a transaction with the seller to either buy or sell an underlying asset at a fixed price on or prior to a fixed date. The fixed price is called the *strike price* and the fixed date is called the *expiration date*. The seller grants this right in exchange for a certain amount of money, called the *option price*. An option is called the *call option* or *put option* depending on whether the buyer is purchasing the right to buy or sell the underlying asset. The simplest option is the *European option* [Wilmott 2006], which can be exercised only on the expiration date. This differs from the *American option* [Wilmott 2006], which can be exercised at any time during the contract lifetime. Both European and American options are called *standard options*.

In the beginning of the 1980s, standard options became more widely understood and their trading volume increased dramatically. Financial institutions began to search for alternative forms of options, known as *exotic options* [Zhang 1998], to meet their new business needs. Among them, two types of options, multi-asset options and multi-exercise options, are particularly relevant to our research.

Multi-asset options are the options written on at least two underlying assets [Zhang 1998]. These underlying assets can be stocks, bonds, currencies, and indices in either the same category or different markets. Several types of multi-asset options are worth mentioning, such as basket options, dual-strike options, rainbow options, paying the best and cash options, and quotient options. Table I provides a brief summary of these multi-asset options, and compares them to standard options and our proposed multi-keyword multi-click ad options (see Section 3) along the following seven dimensions: payoff function, underlying variable, exercise opportunity, early exercise opportunity, strike price and application area. The comparison indicates that our proposed ad options are more complex than previous proposals.

In Table I, it is worth emphasising basket options and dual-strike options. *Basket options* are those options whose payoff is determined by the weighted sum of underlying asset prices [Wilmott 2006]. This structure can be extended to the keyword broad

Table I. Comparison of the Proposed Ad Option and Other Options

Option contract	Payoff function	Underlying variable	Exercise opportunity	Early exercise	Strike price	Application area
n-keyword 1-click ad option (keyword exact or broad match)	$\max\{C_1(t) - F_1, \ldots, C_n(t) - F_n, 0\}$	Multiple	Single	Yes	Multiple	Keywords
n-keyword 1-click ad option (keyword broad match)	$\max\left\{\sum_{i=1}^{k_1} \omega_{1i} C_{1i}(t) - F_1, \ldots, \sum_{i=1}^{k_n} \omega_{ni} C_{ni}(t) - F_n, 0\right\}$	Multiple	Single	Yes	Multiple	Keywords
European standard call option [Wilmott 2006]	$\max\{C(T) - F, 0\}$	Single	Single	No	Single	Equity stock, or index
American standard call option [Wilmott 2006]	$\max\{C(t) - F, 0\}$	Single	Single	Yes	Single	Equity stock, or index
European basket call option [Krekel et al. 2006]	$\max\left\{\sum_{i=1}^{n} \omega_i C_i(T) - F, 0\right\}$	Multiple	Single	No	Single	Index of equity stocks, bonds, or foreign currencies
European dual-strike call option [Zhang 1998]	$\max\{C_1(T) - F_1, C_2(T) - F_2, 0\}$	Double	Single	No	Double	
European rainbow call on max option [Ouwehand and West 2006]	$\max\{\max\{C_1(T), \ldots, C_n(T)\} - F, 0\}$	Multiple	Single	No	Single	Equity stocks, or indexes of equity
European paying the best and cash option [Johnson 1987]	$\max\{C_1(T), C_2(T), F\}$	Double	Single	No	Single	stocks, or bonds, or foreign
European quotient call option [Zhang 1998]	$\max\{C_1(T)/C_2(T) - F, 0\}$	Double	Single	No	Single	currencies

Note: The price of the ith underlying asset/keyword at time t is denoted by $C_i(t)$, where t is a continuous time point in period $[0, T]$ and T is the contract expiration date; if there is only one underlying asset, we denote its price by $C(t)$. The strike/fixed payment price of the ith underlying asset/keyword is denoted by F_i; if there is only one strike price, we denote it by F. The weight of ith asset/keyword in a basket-type option is denoted by ω_i. Note that, in the n-keyword 1-click ad option, ω_{ji} represents the weight of the ith broad matched keyword for the jth candidate keyword, and k_j represents the number of broad matched keywords. Detailed descriptions of notations are provided in Table II.

match setting[1], in which the weights are the probabilities that subphrases occur in search queries. *Dual-strike options* are options with two different strike prices for two different underlying assets [Zhang 1998]. One simple version of our proposed ad options is a dual-strike call option, which allows advertisers to switch between their targeted two keywords during the contract lifetime. However, in sponsored search, the number of candidate keywords to choose from is usually more than two, thus the two keywords are extended to higher dimensions. In addition, as an advertiser usually needs more than a single click for guaranteed delivery, the dual-strike call option is extended to a multi-exercise option.

Multi-exercise options are a generalisation of American options, which provide a buyer with more than one exercise right and sometimes control over one or more other variables [Villinski 2004], for example, the amount of the underlying asset exercised in certain time periods. Multi-exercise options have become more prevalent over the past decade, particularly in the energy industry, such as electricity swing options and water options. Contributors to the multi-exercise options include Deng [2000], Deng and Oren [2006], Clewlow and Strickland [2000], Villinski [2004], Weron [2006], Marshall et al. [2011], and Marshall [2012]. Their work is not further discussed here as our proposed ad option is a simple example of multi-exercise options. Compared to the energy industry, the multi-exercise opportunity in sponsored search is more flexible. Advertisers are allowed to exercise options at any time in the option lifetime, that is, the exercise time is not prespecified, and no minimum number of clicks is required for each exercise. Therefore, there is no penalty fee if the advertiser does not exercise the minimum clicks. In addition, there is no transaction fee for ad options in sponsored search.

Motivated by an attempt to model the fluctuations of asset prices, Brownian motion (i.e., the continuous-time random walk process [Shreve 2004]) was first introduced by Bachelier [1900] to price an option. However, the impact of his work was not recognised by financial community for many years. Sixty-five years later, Samuelson [1965] replaced Bachelier's assumptions on asset price with a geometric form, called the *geometric Brownian motion (GBM)*. In the GBM model, the proportional price changes are exponentially generated by a Brownian motion. While the GBM model is not appropriate for all financial assets in all market conditions, it remains the reference model against which any alternative dynamics are judged.

The research of Samuelson highly affected Black and Scholes [1973] and Merton [1973], who then examined the option pricing based on a GBM. They constructed a portfolio from risky and riskless underlying assets to replicate the value of a European option. Risky assets can be stocks, foreign currencies, indices, and so on; riskless assets can be bonds. Once the risky part of the replicated portfolio is estimated, the option value can be obtained accordingly. The pricing methods proposed by Black and Scholes [1973] and Merton [1973] were based on the assumption that investors on the market cannot obtain arbitrage. Therefore, the replicated portfolio is treated as a self-adjusting process whose least expectation of returns increases at the same speed as the constant bank interest rate. If considering the constant bank interest rate as a discount factor, the discounted value of the replicated portfolio would be a martingale [Björk 2009], whose probability measure is called the *risk-neutral probability measure*. Since a closed-form pricing formula can be obtained from the settings of Black and Scholes

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[1]The keyword match type setting helps the search engine to control which searches can trigger an advertiser's ad. Under the *exact match setting*, the advertiser's ad may show on searches that are an exact term and close variations of that exact term. Under the *broad match setting*, the advertiser's ad may show on searches that include misspellings, synonyms, other relevant variations, and related searches. For further details, see https://support.google.com/adwords/.

[1973] and Merton [1973], we normally call their work the *Black-Scholes-Merton (BSM) option pricing formula*. The BSM option pricing formula spurred research in this field. Various numerical procedures then appeared, including lattice methods, finite difference methods, and Monte Carlo methods. These numerical procedures are capable of evaluating more complex options when the closed-form solution does not exist. In this article, the Monte Carlo method that we discussed earlier can quickly price an ad option when the number of candidate keywords is larger than two.

2.2. Guaranteed Advertising Deliveries

Guaranteed contracts appeared in the early stages of online advertising (particularly in display advertising). They were mostly negotiated by advertisers and publishers[2] privately [Edelman et al. 2007]. Each negotiation contains an amount of needed display impressions over a certain period of time and a prespecified guaranteed price. Hence, in discussing guaranteed deliveries, the following issues must be considered: allocation and pricing. Many studies discussed these two issues separately. Allocation models is reviewed first, then the pricing models.

Feldman et al. [2009] studied an ad selection algorithm for a publisher whose objective is not only to fulfill the guaranteed contracts but also to deliver the well-targeted display impressions to advertisers. This research was more relevant to a service matching problem. The allocation of impressions between the guaranteed and nonguaranteed channels was first discussed by Ghosh et al. [2009], in which a publisher was considered to act as a bidder who bids for guaranteed contracts. This modelling setting was reasonably good as the publisher acts as a bidder who would allocate impressions to online auctions only when other winning bids are high enough. Balseiro et al. [2011] investigated the same allocation problem but used some stochastic control models. In their model, for a given price of an impression, the publisher can decide whether to send it to ad exchanges or assign it to an advertiser with a fixed reserve price. The decision-making process aims to maximise the expected total revenue. Roels and Fridgeirsdottir [2009] proposed a similar allocation framework to Balseiro et al. [2011], where the publisher can dynamically select which guaranteed buy requests to accept and to deliver the guaranteed impressions accordingly. However, compared to Balseiro et al. [2011], the uncertainty in advertisers' buy requests and the traffic of a website were explicitly modelled under the revenue maximisation objective. Recently, a lightweight allocation framework was proposed by Bharadwaj et al. [2012]. They used a simple greedy algorithm to simplify the computations of revenue maximisation.

Two algorithms for pricing the guaranteed display contracts were discussed by Bharadwaj et al. [2012]. Each contract has a large number of impressions and the proposed algorithms solved the revenue optimisation problem for the given number of user visits (i.e., the demand level). However, their work did not consider the auction effects on the contract pricing, and the developed algorithms were purely based on the statistics of users' visits.

Consider the case in which the online advertising market is bouyant (i.e., the winning payment prices for specific ad slots from online auctions increase) and nonguaranteed selling becomes more profitable for publishers. In this case, they may want to cancel the sold guaranteed contracts before the time that the targeted impressions will be created. Online auctions with cancellations are discussed by Babaioff et al. [2009] and Constantin et al. [2009]. They both consider a design in which a publisher can cancel the sold guaranteed contracts but needs to pay a penalty to the advertisers. The proposed auctions with cancellations exhibit interesting economic properties, such as allocative efficiency and equilibrium solution. However, there may exist speculators who pursue

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[2]Publishers are sellers in display advertising.

the cancellation penalty only. In fact, the discussed cancellation penalty is very similar to overselling of flight tickets [Talluri and van Ryzin 2005].

Salomatin et al. [2012] studied a framework of guaranteed deliveries for sponsored search, under which advertisers are able to send their guaranteed requests to a search engine. Each guaranteed request includes the needed number of clicks and the ad budget. The search engine then decides guaranteed deliveries according to search queries and available positions. Since the allocation decision is based on the joint revenue maximisation from guaranteed deliveries and keyword auctions, some advertisers may not receive all their demanded clicks. In such cases, the search engine pays a penalty. However, advertisers still have less control of the ad exposure time and the position of the ad. In addition, with the number of guaranteed advertisers increasing, it is less likely that advertisers can meet their business needs with such a mechanism.

The concept of ad option was initially introduced by Moon and Kwon [2010] (even though Meinl and Blau [2009] discussed the possibility of Web service derivatives, their proposal was not intended for online advertising). Moon and Kwon [2010] proposed that the ad option buyer can be guaranteed the right to choose the minimum payment between cost-per-mille (CPM) and CPC once click-through rate (CTR) is realized. This option structure was similar to a *paying the worst and cash option* [Zhang 1998]. In addition, Moon and Kwon [2010] suggested option pricing under the framework of a Nash bargaining game. Simply, they considered two utility functions: one for the advertiser and one for the publisher. The objective function is the product of these two utilities, and each utility function is restricted by a negotiation power. Therefore, the option price is the optimal solution which maximises the negotiated join utility. Another ad option was discussed by Wang and Chen [2012] (and later Chen and Wang [2015]) for display advertising. The option allows its buyer to select a preferred payment scheme (either CPM or CPC) for the fixed payment. For example, an advertiser can choose to pay a fixed CPC for targeted display impressions. They discussed the lattice methods for option pricing and investigated the stochastic volatility (SV) model for the cases in which the GBM assumption is not valid empirically. However, their work was limited to an univariate case as the SV model cannot be easily extended to multiple variables based on the lattice framework.

3. MULTI-KEYWORD MULTI-CLICK AD OPTIONS

In this section, we introduce how a multi-keyword multi-click ad option works, discuss the option pricing methods, and provide an analysis of the search engine's revenue.

3.1. Guaranteed Delivery in Sponsored Search via Ad Options

We use the following example to illustrate our idea. Suppose that a computer science department creates a new master degree programme on 'Web Science and Big Data Analytics' and is interested in an advertising campaign based around relevant search terms such as 'MSc Web Science', 'MSc Big Data Analytics' and 'Data Mining'. The campaign is to start immediately and last for 3 months. The goal is to generate at least 1000 clicks on the ad, which directs users to the homepage of this new master programme. The department (i.e., advertiser) does not know how the clicks will be distributed among the candidate keywords, nor how much the campaign will cost if based on keyword auctions. However, with the ad option, the advertiser can submit a request to the search engine to lock-in the advertising cost. The request consists of the candidate keywords, the overall number of clicks needed, and the duration of the contract. The search engine responds with a price table for the option, as shown in Figure 1. It contains the option price and the fixed CPC for each keyword. The CPCs are fixed, yet different across the candidate keywords. The contract is entered into when the advertiser pays the option price.
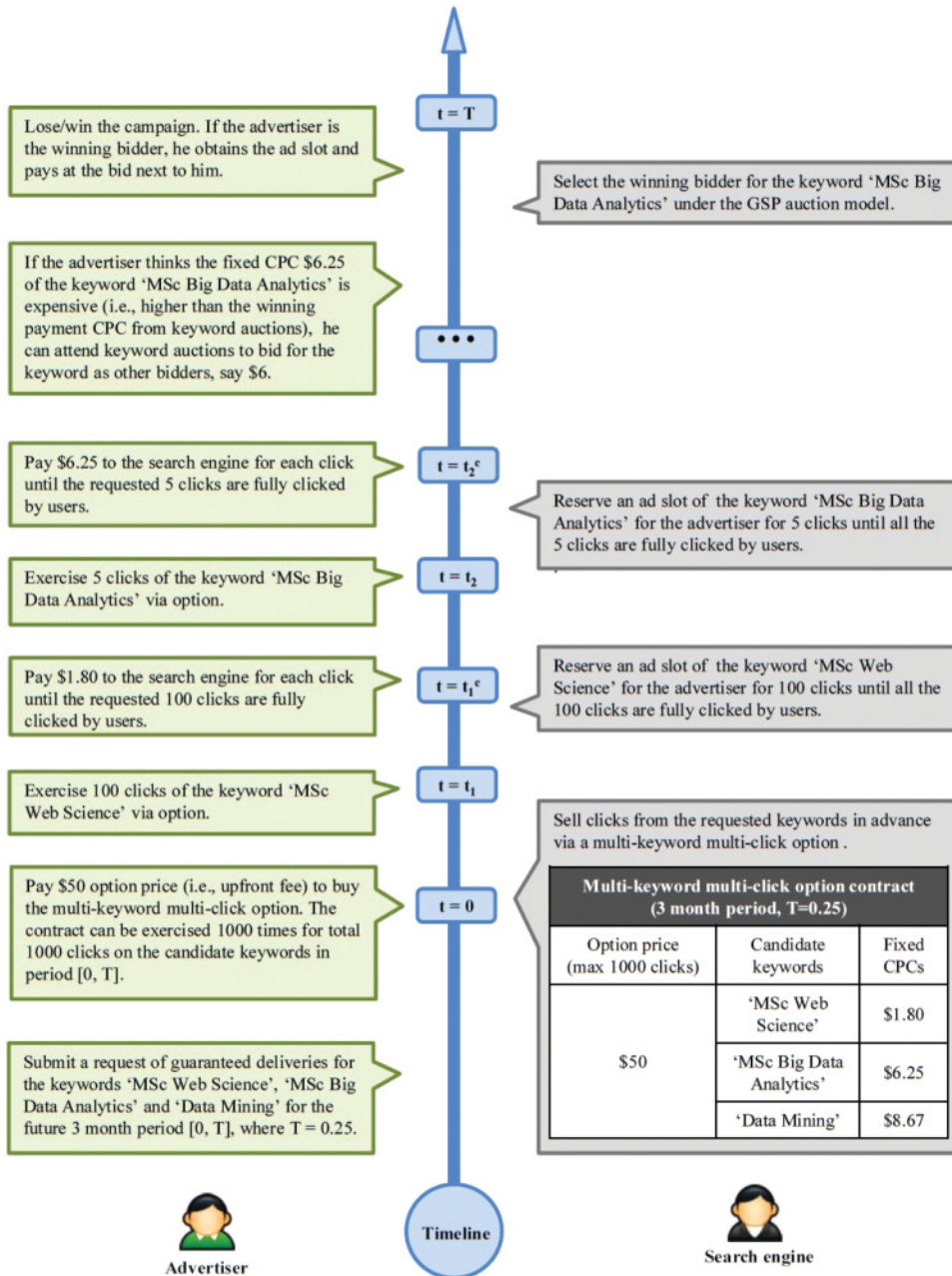
Fig. 1. Schematic view of buying, selling, and exercising a multi-keyword multi-click ad option in sponsored search.

During the contract period $[0, T]$, where T represents the contract expiration date (and is 3 months in this example), the advertiser has the right, at any time, to exercise portions of the contract, for example, to buy a requested number of clicks for a specific keyword. This right expires after time T or when the total number of clicks have been purchased, whichever is sooner. For example, at time $t_1 \leq T$, the advertiser may exercise the right for 100 clicks on the keyword 'MSc Web Science'. After receiving the exercise request, the search engine immediately reserves an ad slot for the keyword for the advertiser until the ad is clicked on 100 times. In our current design, the search engine decides which rank position the ad should be displayed as long as the required number of clicks is fulfilled — we assume that there are adequate search impressions within the period. It is also possible to generalise the study in this article and define a rank-specific option in which all the parameters (CPCs, option prices, and so on) become rank specific. The advertiser can switch among the candidate keywords and also monitor the keyword auction market. If, for example, the CPC for the keyword 'MSc Web Science' drops below the fixed CPC, then the advertiser may choose to participate in the auction rather than exercise the option for the keyword. If, later in the campaign, the spot price for the keyword 'MSc Web Science' exceeds the fixed CPC, the advertiser can then exercise the option.

Figure 1 illustrates the flexibility of the proposed ad option. Specifically, (i) the advertiser does not have to use the option and can participate in keyword auctions as well, (ii) the advertiser can exercise the option at any time during the contract period, (iii) the advertiser can exercise the option up to the maximum number of clicks, (iv) the advertiser can request any number of clicks in each exercise provided the accumulated number of exercised clicks does not exceed the maximum number, and (v) the advertiser can switch among keywords at each exercise at no additional cost. Of course, this flexibility complicates the pricing of the option, which is discussed next.

3.2. Option Pricing Methods

The proposed multi-keyword multi-click ad option enables advertisers to fix their advertising cost and construct a set of candidate keywords beforehand, yet leave the decision of selecting suitable keywords for matching and the exact timing to place the ad to later. Since the advertiser enjoys great flexibility in sponsored search, there is an intrinsic value associated with an ad option, and the buyer needs to pay an upfront option price first. In the following discussion, we focus on calculating a fair upfront option price for the given option candidate keywords, the current winning payment prices, the volatility of these keywords, the contract period length, the riskless bank interest rate, and the fixed CPCs for candidate keywords. Note that the fixed CPCs are considered as given variables, as they can be set by the search engine after receiving the advertiser's request or be proposed by the advertiser in the advertiser's request. Either case will not affect our valuation of the option. We follow the scenario of the motivating example presented in Figure 1 and consider the search engine sets the fixed CPCs.

Recall that Table I presents two different payoff functions for the proposed ad option. The first payoff function can be used to price an ad option with either the keyword exact or broad match setting, which is determined by the match type of the winning payment prices used. However, if only having the exact match winning payment prices from keyword auctions and the advertiser wants to have an ad option with keyword broad match setting, the second payoff function can be used for option pricing. In the following, we discuss the option pricing based on the first payoff function. The same method can be applied to the second payoff function; for further details, see Section 3.2.4.

Table II. Summary of Notations

Notation	Description
r	Constant continuous-time (riskless) interest rate
T	Option expiration date
t	Continuous time point in $[0, T]$
m	Number of total clicks specified by an ad option
n	Number of total number of keywords specified by an ad option
$\boldsymbol{K}$	Keywords specified by an ad option, $\boldsymbol{K} = \{K_1, \ldots, K_n\}$
$\boldsymbol{F}$	Prespecified fixed CPCs for keywords $\boldsymbol{K}$
$\boldsymbol{C}(t)$	Winning payment CPCs for keywords $\boldsymbol{K}$ from auctions at time t
$V(t, \boldsymbol{C}(t); T, \boldsymbol{F}, m)$	Value of an n-keyword m-click ad option at time t
μ_i	Constant drift of CPC for keyword $K_i, i = 1, \ldots, n$
σ_i	Constant volatility of CPC for keyword $K_i, i = 1, \ldots, n$
$W(t)$	Standard Brownian motion at time t
$\boldsymbol{\Sigma}$	Price correlation matrix, in which ρ_{ij} is the correlation coefficient between keywords K_i and K_j, such that $\rho_{ii} = 1$ and $\rho_{ij} = \rho_{ji}$
$\boldsymbol{M\Sigma M}$	Price covariance matrix, where $\boldsymbol{M}$ is the matrix with σ_i along the diagonal and zeros everywhere else
$\Phi(\boldsymbol{C}(t))$	Payoff function of an ad option at time t
π_0	Option price (i.e., upfront fee) of an ad option
$N(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2
$MVN(\boldsymbol{\mu}, \boldsymbol{M\Sigma M})$	Multivariate normal distribution with mean $\boldsymbol{\mu}$ and variance $\boldsymbol{M\Sigma M}$
$\mathcal{N}[\cdot]$	Cumulative probability distribution of a standard normal distribution

3.2.1. Underlying Stochastic Model. The winning payment CPC of the candidate keyword K_i (for a specific slot/position) at time t is denoted by $C_i(t)$. Its movement can be described by a multivariate GBM [Samuelson 1965]:

$$dC_i(t) = \mu_i C_i(t)dt + \sigma_i C_i(t)dW_i(t), \quad i = 1, \ldots, n, \tag{1}$$

where μ_i and σ_i are constants representing the drift and volatility of the CPC, respectively, and $W_i(t)$ is a standard Brownian motion satisfying the conditions:

$$\mathbb{E}(dW_i(t)) = 0,$$
$$\text{var}(dW_i(t)) = \mathbb{E}(dW_i(t)dW_i(t)) = dt,$$
$$\text{cov}(dW_i(t), dW_j(t)) = \mathbb{E}(dW_i(t)dW_j(t)) = \rho_{ij}dt,$$

where ρ_{ij} is the correlation coefficient between keywords K_i and K_j, such that $\rho_{ii} = 1$ and $\rho_{ij} = \rho_{ji}$. The correlation matrix is denoted by $\boldsymbol{\Sigma}$, so that the covariance matrix is simply $\boldsymbol{M\Sigma M}$, where $\boldsymbol{M}$ is the matrix with the σ_i along the diagonal and zeros everywhere else. For the reader's convenience, detailed descriptions of notations are provided in Table II.

Since the GBM assumption lays the foundation for pricing the proposed ad option, we provide several discussions and investigations of it. In Section 3.2.4, we explain why the GBM assumption is suitable for pricing an ad option in sponsored search, and also highlight its limitations. In Section 4.2, we discuss the estimation of GBM parameters. In Section 4.3, we conduct goodness-of-fit tests with real datasets and track the "errors" of the calculated option price when the GBM assumption is not valid empirically.

3.2.2. Terminal Value Pricing. To simplify the discussion and without loss of generality, the value of an n-keyword m-click ad option can be decomposed as the sum of m independent n-keyword 1-click ad options. If an advertiser buys an ad option at time 0, the

option price π_0 can be expressed as follows:

$$\pi_0 = V(0, \boldsymbol{C}(0); T, \boldsymbol{F}, m) = mV(0, \boldsymbol{C}(0); T, \boldsymbol{F}, 1), \tag{2}$$

where $V(0, \boldsymbol{C}(0); T, \boldsymbol{F}, m)$ represents the option value at time 0.

Our focus now centres on the n-keyword 1-click ad option. Adopting the basic economic setting [Narahari et al. 2009], we assume that an advertiser is risk-neutral. In other words, the advertiser has no preference across the candidate keywords and exercises the option for the keyword which has the maximum difference between its winning payment price and the prespecified fixed price. This difference shows the value of the option because the advertiser is offered the right to move from the auction market to the guaranteed market.

Let us first consider if the advertiser exercises the option at the contract expiration date T, the option payoff can be defined as follows:

$$\Phi(\boldsymbol{C}(T)) = \max\{C_1(T) - F_1, \ldots, C_n(T) - F_n, 0\}. \tag{3}$$

Note that the option payoff in sponsored search does not mean the direct reward, but it measures the difference of advertising cost between the auction market and the guaranteed market. By having Equation (3), we can see if the advertiser would like to exercise the option early by using the backward deduction method. The option value at time $t < T$ is then

$$V(t, \boldsymbol{C}(t); T, \boldsymbol{F}, 1) = \begin{cases} \Phi(\boldsymbol{C}(t)), & \text{if early exercise,} \\ \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)}\Phi(\boldsymbol{C}(T))], & \text{if not early exercise,} \end{cases}$$

where r is the constant riskless bank interest rate and $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ is the conditional expectation with respect to time t under the probability measure $\mathbb{Q}$. As we use the riskless bank interest rate as the discounted factor, the probability measure $\mathbb{Q}$ is also called the *risk-neutral probability measure* [Björk 2009]. Appendix B discusses the rationale for using the riskless bank interest rate and introduces an alternative method of option pricing.

Let us now return to the decision-making problem. If the ad option is exercised early at time t, the option value is equal to its payoff $\Phi(\boldsymbol{C}(t))$. However, if the ad option is not exercised, the option value at time t is equal to the discounted value of the expected payoff at the expiration date T. The comparison between $\Phi(\boldsymbol{C}(t))$ and $\mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)}\Phi(\boldsymbol{C}(T))]$ informs the optimal decision for the advertiser. Since the payoff function defined is convex, we then obtain the following inequality (see Appendix A):

$$\Phi(\boldsymbol{C}(t)) \leq \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)}\Phi(\boldsymbol{C}(T))]. \tag{4}$$

Equation (4) illustrates, to gain the maximum option value, that the advertiser will not exercise the option until its expiration date. Hence, the option price should be computed at the discounted value of the expected payoff from the expiration date T. Together with Equation (2), we can obtain the option pricing formula for the n-keyword m-click ad option:

$$\pi_0 = me^{-rT}\mathbb{E}_0^{\mathbb{Q}}[\Phi(\boldsymbol{C}(T))]. \tag{5}$$

It is worth noting that we rule out arbitrage [Varian 1987] between the auction market and the guaranteed market in option pricing. The concept of arbitrage can be understood as the "free lunch". As a market designer, we need to make sure that everyone obtains something by paying something so that it is fair to both the buy and sell sides. Since we assume that an advertiser is risk-neutral, the riskless bank interest rate can be employed as the benchmark rate to rule out arbitrage. Equation (5) can also

---

ALGORITHM 1: Pricing a Multi-keyword Multi-click Ad Option via Monte Carlo Simulations.

function OptionPricingMC(K, $C(0)$, Σ, M, m, r, T)

    for $k \leftarrow 1$ to $\widetilde{n}$ **do**   # $\widetilde{n}$ is the number of simulations;

        $[z_{1,k}, \ldots, z_{n,k}] \leftarrow$ GeneratingMultivariateNoise($MVN[0, M\Sigma M]$)

        for $i \leftarrow 1$ to n **do**

            $C_{i,k} \leftarrow C_i(0) \exp\left\{(r - \frac{1}{2}\sigma_i^2)T + \sigma_i z_{i,k}\sqrt{T}\right\}.$

        end for

        $G_k \leftarrow \Phi([C_{1,k}, \ldots, C_{n,k}]).$

    end for

    $\pi_0 \leftarrow me^{-rT}\mathbb{E}_0[\Phi(C(T))] \approx me^{-rT}\left(\frac{1}{\widetilde{n}}\sum_{k=1}^{\widetilde{n}} G_k\right).$

    return π_0

end function

Note: A detailed description of notations is provided in Table II.

---

be obtained by constructing an advertising strategy for the advertiser, as discussed in Appendix B.

3.2.3. Solutions. Equation (5) can be expanded in integral form as follows:

$$\pi_0 = me^{-rT}(2\pi T)^{-\frac{n}{2}}|\Sigma|^{-\frac{1}{2}}\left(\prod_{i=1}^{n}\sigma_i\right)^{-1}$$

$$\times \int_0^{\infty}\cdots\int_0^{\infty}\frac{\Phi(\widetilde{C})}{\prod_{i=1}^{n}\widetilde{C}_i}\exp\left\{-\frac{1}{2}\zeta^T\Sigma^{-1}\zeta\right\}d\widetilde{C}, \tag{6}$$

where $\zeta = (\zeta_1, \ldots, \zeta_n)'$, $\zeta_i = \frac{1}{\sigma_i\sqrt{T}}\left(\ln\{\widetilde{C}_i/C_i(0)\} - (r - \frac{\sigma_i^2}{2})T\right)$, and other notations are described in Table II.

Closed-form solutions to Equation (6) can be derived if $n \leq 2$. If $n = 1$, Equation (6) is equivalent to the Black-Scholes-Merton (BSM) pricing formula for a European call option [Black and Scholes 1973; Merton 1973]. If $n = 2$, Equation (6) contains a bivariate normal distribution and the option price can be obtained by employing the pricing formula for a dual-strike European call option [Zhang 1998]. The closed-form solutions are provided in Appendix C.

For $n \geq 3$, taking integrals in Equation (6) is computationally difficult. In such a case, we resort to numerical techniques to approximate the option price. Algorithm 1 illustrates our Monte Carlo method. For $\widetilde{n}$ number of simulations, for each simulation, we generate a vector of multinormal noise, then calculate the CPCs at time T. Equation (4) shows that there is no need to generate the whole paths in each simulation, as we only consider the CPCs on the expiration date in the calculation of option payoff. Hence, by having $\widetilde{n}$ payoffs at time T, the option price π_0 can be approximated numerically. We refer to this as Algorithm 1.

3.2.4. Discussion. The candidate keywords' prices may not follow the GBM assumption empirically because some time series features, such as jumps and volatility clustering, cannot be captured effectively by a GBM [Marathe and Ryan 2005]. However, the GBM model is still a good choice for pricing ad options in sponsored search. First, in our data analysis (see Section 4.3.1), we find that 15.73% of keywords' CPCs satisfy the GBM assumption. Second, for the cases in which the GBM assumption is not valid empirically (see Section 4.3.2), we find that the pricing model is reasonably robust as

the identified arbitrage values in many experimental groups are small. Of course, our dataset might be biased. However, previous research in keyword auctions supports the GBM assumption: Lahaie and Pennock [2007] tested the log-normality of bids on Yahoo! search advertising data and gave the estimated distribution parameters; Ostrovsky and Schwarz [2011] performed experiments based on the log-normal bids on the Yahoo! search advertising platform; Pin and Key [2011] observed random bids from the Microsoft Bing search platform and simulated similar bids based on the log-normal distribution. Since in this research the advertisers' bids are tested across auctions, the winning payment prices (i.e., the second-highest bids from auctions) over time also satisfy the log-normal distribution. Recall that, in the GBM model, the difference between two logarithms of winning payment prices follows a time-dependent normal distribution. If we consider the average daily winning payment price as the underlying variable, these previous works can provide the distribution hypothesis tests to support the GBM assumption in sponsored search. However, for display advertising, the GBM assumption is usually not valid empirically, which has been investigated recently by Chen et al. [2014] and Yuan et al. [2014].

Table I shows that, if only having the exactly matched $\boldsymbol{C}(T)$, we can still construct a broad match structure for the option. Similar to Equation (3), the option payoff function on time T can be defined as follows:

$$\Phi(\boldsymbol{C}(T)) = \max\left\{\sum_{i=1}^{k_1}\omega_{1i}C_{1i}(T) - F_1, \ldots, \sum_{i=1}^{k_n}\omega_{ni}C_{ni}(T) - F_n, 0\right\}, \qquad (7)$$

where ω_{ji} is the probability that the ith broad matched keyword (i.e., the subphrase occurs in search queries) for the keyword K_j, and k_j represents the number of broad matched keywords. Equation (1) can be still used to model the underlying CPCs' movement and the option price π_0 can be directly calculated by Algorithm 1.

3.3. Revenue Analysis for Search Engine

The proposed ad option can be considered as "insurance" for an advertiser. The advertiser needs to pay the upfront option price, which contributes to the search engine's revenue. In the following discussion, we analyse the effect of an ad option on the search engine's revenue. We provide a functional analysis for the 1-keyword 1-click ad option in this section and leave the empirical investigation of the n-keyword cases to Section 4.

Let $D(F)$ be the difference between the expected revenue from an ad option and the expected revenue from only keyword auctions; we then have

$$D(F) = \underbrace{\left(C(0)\mathcal{N}[\zeta_1] - e^{-rT}F\mathcal{N}[\zeta_2] + e^{-rT}F\right)\mathbb{P}\left(\mathbb{E}_0^{\mathbb{Q}}[C(T)] \geq F\right)}_{= \text{ Discounted value of expected revenue from option if }\mathbb{E}_0^{\mathbb{Q}}[C(T)]\geq F}$$

$$+ \underbrace{\left(C(0)\mathcal{N}[\zeta_1] - e^{-rT}F\mathcal{N}[\zeta_2] + e^{-rT}\mathbb{E}_0^{\mathbb{Q}}[C(T)]\right)\mathbb{P}\left(\mathbb{E}_0^{\mathbb{Q}}[C(T)] < F\right)}_{= \text{ Discounted value of expected revenue from option if }\mathbb{E}_0^{\mathbb{Q}}[C(T)]<F}$$

$$- \underbrace{e^{-rT}\mathbb{E}_0^{\mathbb{Q}}[C(T)]}_{= \text{ Discounted value of expected revenue from auction}}.$$

$$= C(0)\mathcal{N}[\zeta_1] - e^{-rT}F\mathcal{N}[\zeta_2] - e^{-rT}\left(\mathbb{E}_0^{\mathbb{Q}}[C(T)] - F\right) \times \mathbb{P}\left(\mathbb{E}_0^{\mathbb{Q}}[C(T)] \geq F\right), \qquad (8)$$

where $\mathcal{N}[\cdot]$ represents the cumulative probability of a standard normal distribution.

Let us consider the boundary values first. If $F = 0$, the option price π_0 achieves its maximum value $e^{-rT}\mathbb{E}_0^{\mathbb{Q}}[C(T)]$; therefore, $D(F) \to 0$. If $\pi_0 = 0$, the fixed CPC F is as large as possible, and $\mathbb{P}(\mathbb{E}_0^{\mathbb{Q}}[C(T)] \geq F) \to 0$ and $D(F) \to 0$. Since

$$\ln\{C(T)/C(0)\} \sim N\big((r - \sigma^2/2)T, \sigma^2 T\big),$$

we can have

$$
\begin{aligned}
\mathbb{P}\big(\mathbb{E}_0^{\mathbb{Q}}[C(T)] \geq F\big) &= \mathbb{P}\left(C(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)T + \frac{1}{2}\sigma^2 T\right\} \geq F\right) \\
&= \mathbb{P}\left(\ln\{C(T)/C(0)\} + \ln\{F/C(T)\} - rT \leq 0\right) \\
&= \mathbb{P}\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\{C(T)/C(0)\} - \left(r - \frac{1}{2}\sigma^2\right)T\right)\right) \\
&\leq \frac{1}{\sigma\sqrt{T}}\left(\ln\{C(0)/F\} + rT + \sigma W(T)\right)\bigg) = \mathscr{N}[\zeta^*], \quad (9)
\end{aligned}
$$

where $\zeta^* = \frac{1}{\sigma\sqrt{T}}(\ln\{C(0)/F\} + rT + \sigma W(T)) = \zeta_1 - (\frac{1}{2}\sigma\sqrt{T} - \frac{1}{\sqrt{T}}W(T)) = \zeta_2 + (\frac{1}{2}\sigma\sqrt{T} + \frac{1}{\sqrt{T}}W(T))$. Since $\mathbb{E}[W(T)] = 0$ and $\mathrm{var}[W(T)] = T$, then $\zeta_2 \leq \zeta^* \leq \zeta_1$ with 67% probability if $\sigma^2 T \geq 4$. Then

$$
\begin{aligned}
D(F) &= C(0)\mathscr{N}[\zeta_1] - e^{-rT}F\mathscr{N}[\zeta_2] - e^{-rT}\big(\mathbb{E}_0^{\mathbb{Q}}[C(T)] - F\big)\mathscr{N}[\zeta^*] \\
&= C(0)(\mathscr{N}[\zeta_1] - \mathscr{N}[\zeta^*]) + e^{-rT}F(\mathscr{N}[\zeta^*] - \mathscr{N}[\zeta_2]) \geq 0, \quad (10)
\end{aligned}
$$

suggesting that the search engine can have an increased expected revenue if the search engine sells the click via an option rather than through an auction. Taking the derivative of $D(F)$ with respect to F and assigning its value to zero, we have

$$
\begin{aligned}
\frac{\partial D(F)}{\partial F} &= C(0)\frac{\partial \mathscr{N}[\zeta_1]}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial F} - e^{-rT}\mathscr{N}[\zeta_2] - e^{-rT}F\frac{\partial \mathscr{N}[\zeta_2]}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial F} \\
&\quad - e^{-rT}\big(\mathbb{E}_0^{\mathbb{Q}}[C(T)] - F\big)\frac{\partial \mathbb{P}\big(\mathbb{E}_0^{\mathbb{Q}}[C(T)] \geq F\big)}{\partial F} + e^{-rT}\mathbb{P}\big(\mathbb{E}_0^{\mathbb{Q}}[C(T)] \geq F\big) = 0. \quad (11)
\end{aligned}
$$

Since $\partial \mathscr{N}(x)/\partial x = \frac{1}{\sqrt{2\pi_0}}e^{-\frac{1}{2}x^2}$, the following equation holds:

$$\frac{\partial \mathscr{N}[\zeta_2]}{\partial \zeta_2}\bigg/\frac{\partial \mathscr{N}[\zeta_1]}{\partial \zeta_1} = \exp\left\{\frac{1}{2}(\zeta_1^2 - \zeta_2^2)\right\} = \frac{C(0)e^{rT}}{F}. \quad (12)$$

Taking the derivative of ζ_1 and ζ_2 with respect to F gives

$$\frac{\partial \zeta_1}{\partial F} = \frac{\partial \frac{1}{\sigma\sqrt{T}}\big(\ln\{C(0)/F\} + (r + \frac{1}{2}\sigma^2)T\big)}{\partial F} = -\frac{1}{F\sigma\sqrt{T}}, \quad (13)$$

$$\frac{\partial \zeta_2}{\partial F} = \frac{\partial \zeta_1}{\partial F} - \frac{\partial \sigma\sqrt{T}}{\partial F} = -\frac{1}{F\sigma\sqrt{T}}. \quad (14)$$

Table III. Overview of Experimental Settings of Data

Market	Group	Training set (31 days)	Deve&test set (31 days)
US	1	25/01/2012–24/02/2012	24/02/2012–25/03/2012
	2	30/03/2012–29/04/2012	29/04/2012–31/05/2012
	3	10/06/2012–12/07/2012	12/07/2012–17/08/2012
	4	10/11/2012–11/12/2012	11/12/2012–10/01/2013
UK	1	25/01/2012–24/02/2012	24/02/2012–25/03/2012
	2	30/03/2012–29/04/2012	29/04/2012–31/05/2012
	3	12/06/2012–13/07/2012	13/07/2012–19/08/2012
	4	18/10/2012–22/11/2012	22/11/2012–24/12/2012

and $D(F)$ achieves its maximum or minimum value at $F = \mathbb{E}_0^{\mathbb{Q}}[C(T)]$. Further, taking the second derivative of $D(F)$ with respect to $F = \mathbb{E}_0^{\mathbb{Q}}[C(T)]$ gives

$$\frac{\partial^2 D(F)}{\partial F^2} = \frac{\partial \mathbb{P}(\mathbb{E}_0^{\mathbb{Q}}[C(T)] \geq F)}{\partial F} = \frac{\partial \mathcal{N}[\zeta_2]}{\partial \zeta_2} \frac{\partial \zeta_2}{\partial F} = -\frac{1}{\sqrt{2\pi_0}} e^{-\frac{1}{2}\zeta_2^2} \frac{1}{F\sigma\sqrt{T}} < 0.$$

Hence, if the fixed CPC is set the same as the estimated spot CPC on the contract expiration date (i.e., $F = \mathbb{E}_0^{\mathbb{Q}}[C(T)]$), the search engine can increase its profit.

4. EXPERIMENTS

In this section, we describe our data and experimental settings, conduct assumption and fairness tests, and investigate the option's effects on the search engine's revenue.

4.1. Data and Experimental Design

The data used in the experiments is collected from Google AdWords by using its Traffic Estimation Service [Yuan and Wang 2012]: when an advertiser submits targeted keywords, budget, and other settings to Google, the Traffic Estimation Service will return a list of data values, including the estimated CPCs, clicks, global impressions, local impressions, and position. These values are recorded for the period from 26/11/2011 to 14/01/2013, for a total of 557 keywords in the US and UK markets. Note that, in the data, 21 keywords have missing values and 115 keywords' CPCs are all 0.

For each market, as illustrated in Table III, we split the data into 4 experimental groups. Each group has one training, one development, and one test set. The training set is used to: (i) select the keywords with nonzero CPCs; (ii) test the statistical properties of the underlying dynamic, and estimate the model parameters. We then price ad options and simulate the corresponding buying and selling transactions in the development set. Finally, the test set is used as the baseline to examine the priced ad options.

4.2. Parameter Estimation and Option Pricing

The GBM parameters are estimated by using the method suggested by Wilmott [2006]. Specifically, for the keyword K_i, the volatility σ_i is the sample standard deviation of change rates of log CPCs and the correlation ρ_{ij} is given by

$$\rho_{ij} = \frac{\sum_{k=1}^{\tilde{m}} (y_i(k) - \bar{y}_i)(y_j(k) - \bar{y}_j)}{\sqrt{\sum_{k=1}^{\tilde{m}} (y_i(k) - \bar{y}_i)^2 \sum_{k=1}^{\tilde{m}} (y_j(k) - \bar{y}_j)^2}}, \tag{15}$$

where $\tilde{m}$ is the size of training data and $y_i(t_k)$ is the kth change rate of log CPCs.
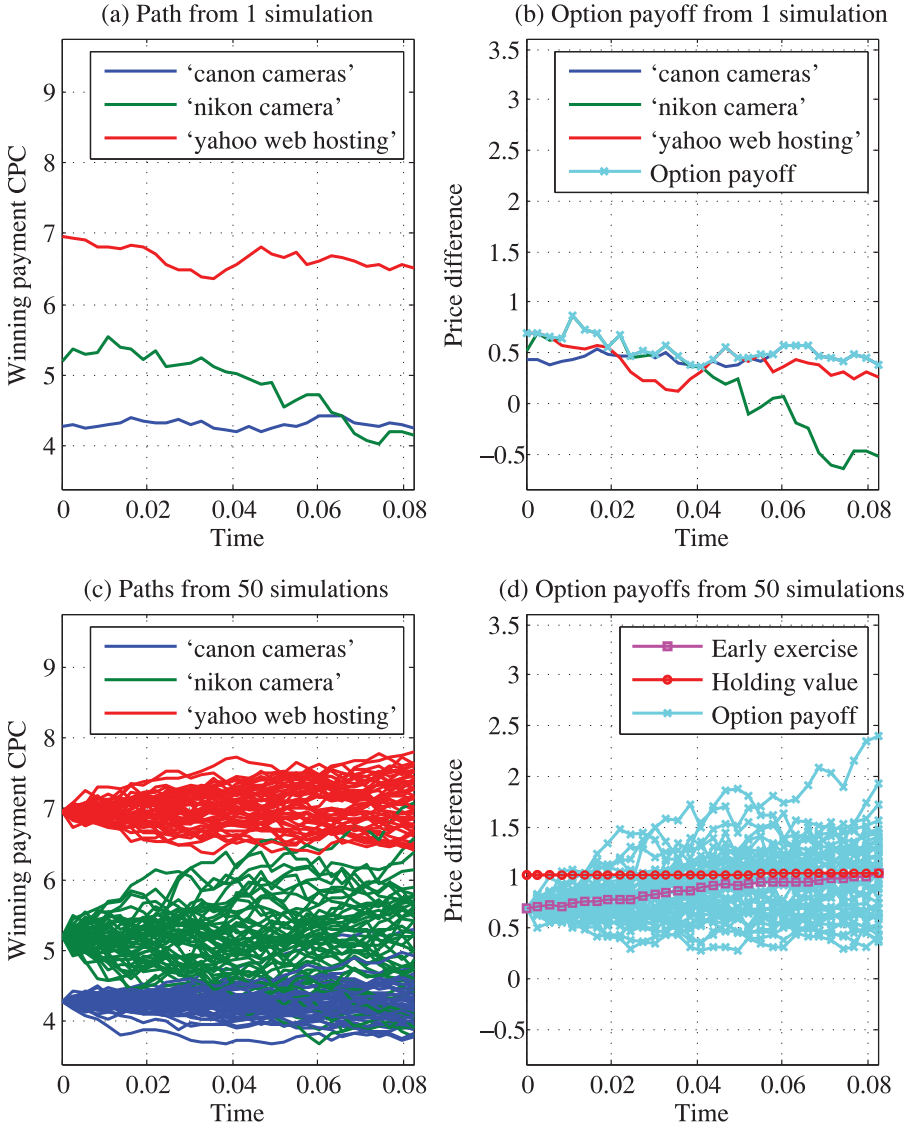
Fig. 2. Empirical example of pricing a 3-keyword 1-click ad option via the Monte Carlo method, where K_1 = 'canon cameras', K_2 = 'nikon camera ', K_3 = 'yahoo web hosting', $F_1 = 3.8505$, $F_2 = 4.6704$, and $F_3 = 6.2520$.

Figure 2 illustrates an empirical example, in which the candidate keywords are

$$\boldsymbol{K} = \left\{ \begin{array}{c} K_1 \\ K_2 \\ K_3 \end{array} \right\} = \left\{ \begin{array}{c} \text{'canon cameras'} \\ \text{'nikon camera'} \\ \text{'yahoo web hosting'} \end{array} \right\},$$

and the estimated model parameters are

$$\boldsymbol{\sigma} = \begin{pmatrix} 0.2263 \\ 0.4521 \\ 0.2136 \end{pmatrix}, \ \boldsymbol{\Sigma} = \begin{pmatrix} 1.0000 & 0.2341 & 0.0242 \\ 0.2341 & 1.0000 & -0.0540 \\ 0.0242 & -0.0540 & 1.0000 \end{pmatrix}.$$

Note that a high contextual relevance of keywords normally means that they have a high substitutional degree to each other, such as 'canon cameras' and 'nikon camera', whose CPCs move in the same direction with correlation 0.2341. The other keyword 'yahoo web hosting' is contextually less relevant to the former keywords and also has very low price correlations to them. The example also shows that the contextual relevance of keywords has an impact on their CPC movement.

Based on the estimated parameters, we draw a sample of simulated paths of a 3-dimensional GBM in Figure 2(a) for 31 days (in which the x-axis is expressed in terms of year value). Recall that the option payoff at any time t in the contract lifetime is $\max\{C_1(t) - F_1, \ldots, C_n(t) - F_n, 0\}$. In Figure 2(b), we plot the price difference between the spot CPC and the fixed CPC of each candidate keyword (i.e., $C_i(t) - F_i, i = 1, \ldots, n$) and also indicate the corresponding option daily payoffs (shown by the cyan curve). It suggests that switching among keywords would help the advertiser to maximise the benefits of the ad option. Repeating these simulations 50 times generates 50 simulated vales of each keyword for each day, as shown in Figure 2(c). We then calculate 50 option payoffs and their daily mean values to obtain the final option price, as shown in Figure 2(d).

To examine the fairness (i.e., no-arbitrage) of the calculated option price, we can construct a riskless value difference process by delta hedging $\partial V / \partial C_j$ (see Appendix B) and check if any arbitrage exists [Wilmott 2006]. The hedging delta of the 1-keyword 1-click ad option can be calculated as follows:

$$\frac{\partial V}{\partial C} = \mathcal{N}\left[\frac{1}{\sigma\sqrt{T}}\left(\ln\left\{\frac{C(0)}{F}\right\} + \left(r + \frac{\sigma^2}{2}\right)T\right)\right]. \tag{16}$$

For the n-keyword 1-click option, the hedging delta of each keyword can be computed by the Monte Carlo method, i.e., $\partial V/\partial C_i = \mathbb{E}^{\mathbb{Q}}[\partial V(T, C(T))/\partial C_i(T)]$. According to Appendix B, we can define the 31-day growth rate of the value difference process as $\widetilde{\gamma} = (\Pi(t_{31}) - \Pi(t_0))/\Pi(t_0)$, and compare $\widetilde{\gamma}$ to the riskless bank interest rate $r = 5\%$ (equivalent to $\widetilde{r} = 4.12\%$ per 31 days return[3]). The arbitrage detection criterion is

$$|\widetilde{\gamma} - \widetilde{r}| \leq \varepsilon ? \textbf{ arbitrage does not exist } : \textbf{ arbitrage exists}, \tag{17}$$

where the notation ε is the model variation threshold (and we set $\varepsilon = 5\%$ in experiments). Hence, a positive $\widetilde{\gamma} - \widetilde{r}$ means that the advertiser who buys an option can obtain arbitrage while a negative $\widetilde{\gamma} - \widetilde{r}$ indicates the case of making arbitrage by selling an option. Then, the identified arbitrage α is defined as the excess return, that is,

$$\alpha = \begin{cases} \widetilde{\gamma} - (\widetilde{r} - \varepsilon), & \text{if } \widetilde{\gamma} < \widetilde{r} - \varepsilon, \\ \widetilde{\gamma} - (\widetilde{r} + \varepsilon), & \text{if } \widetilde{\gamma} > \widetilde{r} + \varepsilon. \end{cases} \tag{18}$$

Table IV presents the overall results of our arbitrage test based on the GBM model. We generate paths for candidate keywords with 100 simulations and examine the options price using delta hedging. There are 99.76% (1-keyword), 93.06% (2-keyword), and 92.71% (3-keyword) options fairly priced. Only a small number of options exhibits arbitrage and most of the mean arbitrage values lie within 5%, such as shown in Figure 3. The existence of small arbitrage may be due to two reasons. First, the stability of process simulations in both option pricing and arbitrage test. Second, the candidate keywords are randomly selected for the 2-keyword and 3-keyword options. The significant differences in the absolute prices of these keywords can generate a large variation of calculated option payoffs, which then trigger arbitrage.

---

[3]The relationship between the continuous compounding r and the return per 31 days $\widetilde{r}$ is: $1 + \widetilde{r} = e^{r \times 30/365}$ [Hull 2009].

Table IV. Test of Arbitrage for Ad Options Based on a GBM

		US market			UK market		
n	Group	N	$\mathbb{P}(\alpha)$	$\mathbb{E}[\alpha]$	N	$\mathbb{P}(\alpha)$	$\mathbb{E}[\alpha]$
1	1	94	0.00%	0.00%	76	0.00%	0.00%
	2	64	0.00%	0.00%	45	0.00%	0.00%
	3	94	1.06%	0.75%	87	0.00%	0.00%
	4	112	0.89%	−0.37%	53	0.00%	0.00%
2	1	47	4.26%	1.63%	38	0.00%	0.00%
	2	32	9.38%	0.42%	22	4.55%	13.41%
	3	47	4.26%	0.84%	43	4.65%	0.82%
	4	56	5.36%	3.44%	26	23.08%	−6.22%
3	1	31	0.00%	0.00%	25	4.00%	0.00%
	2	21	4.76%	−1.38%	15	0.00%	0.00%
	3	31	0.00%	0.00%	29	3.45%	−1.12%
	4	37	10.81%	3.87%	17	35.29%	−2.54%

Note: n is the number of candidate keywords, N is the number of options priced in a group, $\mathbb{P}(\alpha)$ is percentage of options in a group with identified arbitrage, and the $\mathbb{E}[\alpha]$ is the average arbitrage value of the options, where the arbitrage α is defined by Equation (18) and the riskless bank interest rate $r = 5\%$.
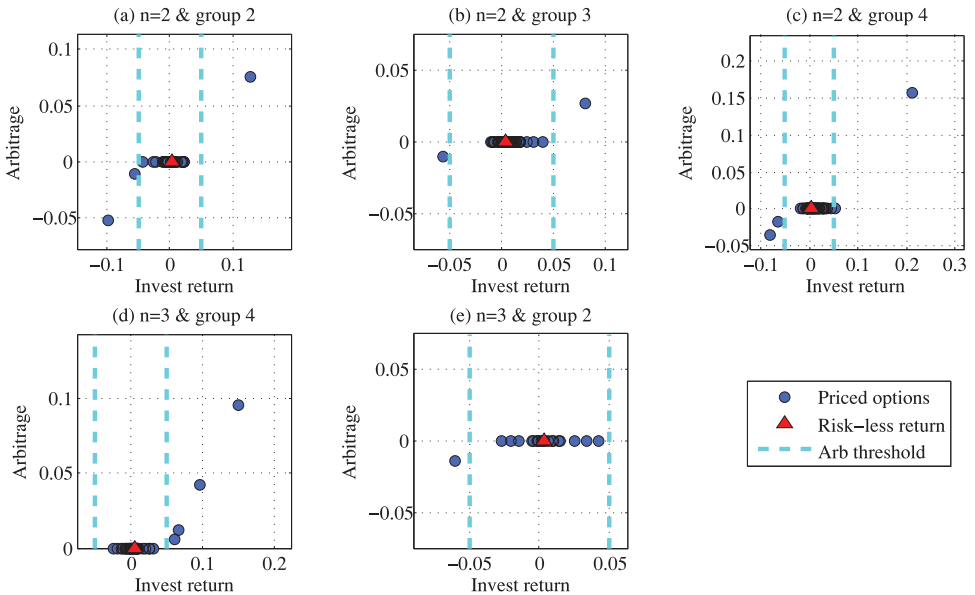

Fig. 3. Empirical example of arbitrage analysis based on GBM for the US market.

4.3. Model Validation and Robustness Test

We now examine the GBM assumption and investigate if arbitrage exists when the candidate keywords in an option do not follow a GBM.

4.3.1. Checking the Underlying GBM Assumption. Two validation conditions of the GBM model are tested [Marathe and Ryan 2005]: (i) the normality of change rates of log CPCs and (ii) the independence from previous data. Normality can be either checked graphically by histogram/Q-Q plot or verified statistically by the Shapiro-Wilk test [Shapiro and Wilk 1965]. To examine independence, we employ the autocorrelation function
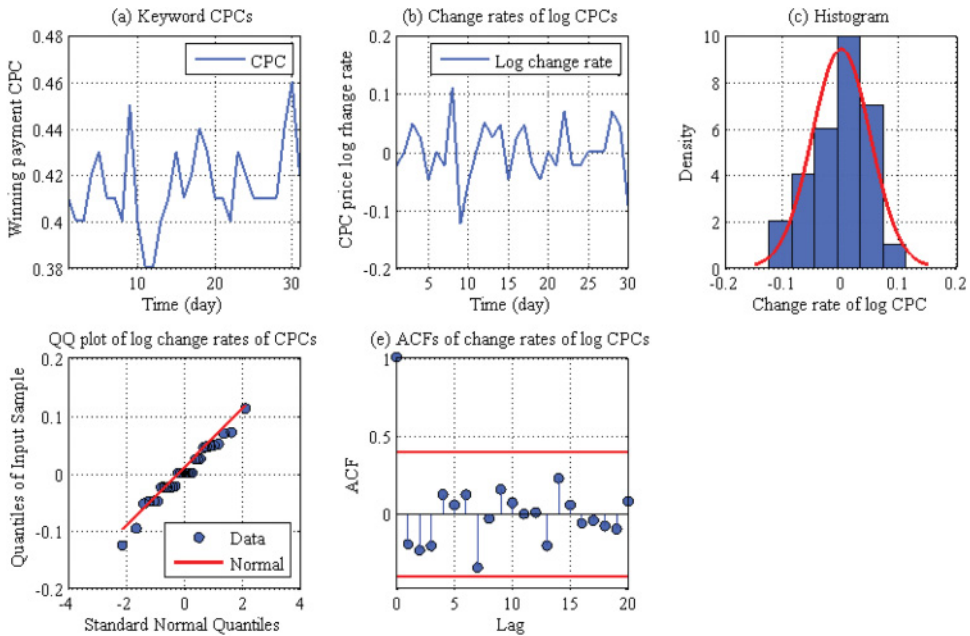
Fig. 4. Empirical example of checking the GBM assumption for the keyword 'canon 5d', in which the Shapiro-Wilk test has a *p*-value of 0.3712 and the Ljung-Box test has a *p*-value of 0.4555.

(ACF) [Tsay 2005] and the Ljung-Box statistic [Ljung and Box 1978]. Figure 4 provides an empirical example of the keyword 'canon 5d'. Figures 4(a) and 4(b) exhibit the movement of CPCs and log change rates while Figures 4(c) and 4(d) show that the stated two conditions are satisfied in this case.

We check the discussed two conditions with the training data. As shown in Figure 5, there are 14.25% and 17.20% of keywords in US and UK markets, respectively, that satisfy the GBM assumption. Thus, 15.73% of keywords can be effectively priced into an option based on a GBM. It is worth mentioning that not all keywords follow a GBM. Next, we examine the robustness of pricing models and investigate the arbitrage based on non-GMB models.

4.3.2. Examining Arbitrage for Non-GBM Dynamics. Several popular stochastic processes (together with the real data) are tested to check the arbitrage in option pricing. Table V shows the candidate models. Each model can capture certain features of time-series data, such as mean reversion, constant volatility, and square root volatility [Hull 2009]. The arbitrage tests here are slightly different from those of the GBM model. We estimate the model parameters from the actual data in the test sets instead of the learning sets and treat the actual data as one single path of each model. Hence, the simulated data has the same drift, volatility, and correlations as the test data. We are now able to examine the arbitrage multiple times when the real-world environment does not follow a GBM. Also, for the candidate models, hypothesis tests are used to check if the simulated path and actual data come from a same distribution. These tests include the Wilcoxon test [Wilcoxon 1945], Ansari-Bradley test [Mood et al. 1974] and Two-sample Kolmogorov-Smirnov test [Justel et al. 1997]. Figure 6 summarises the results of models' goodness-of-fit tests, in which the y-axis represents the mean percentage of simulated paths not rejected by the hypothesis tests. Even though the three tests give different absolute percentages, the dynamics' performance is similar and consistent:
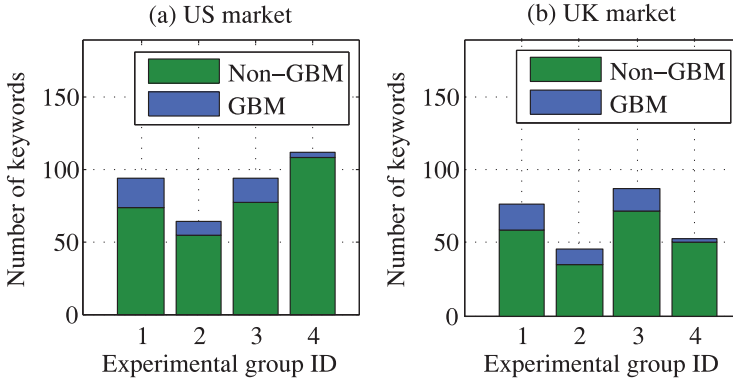
Fig. 5.   Overview of checking the GBM assumption for all keywords of experimental groups.

Table V. Tested non-GBM Dynamics

Dynamic	Stochastic differential equation (SDE)
Constant elasticity of variance (CEV) model [Cox and Ross 1976]	$dC_i(t) = \mu_i C_i(t)dt + \sigma_i(C_i(t))^{1/2}dW_i(t)$
Mean-reverting drift (MRD) model [Wilmott 2006]	$dC_i(t) = k_i(\mu_i - C_i(t))dt + \sigma_i(C_i(t))^{1/2}dW_i(t)$
Cox-Ingersoll-Ross (CIR) model [Cox et al. 1985]	$dC_i(t) = k_i(\mu_i - C_i(t))dt + (\sigma_i)^{1/2}C_i(t)dW_i(t)$
Hull-White/Vasicek (HWV) model [Hull and White 1990]	$dC_i(t) = k_i(\mu_i - C_i(t))dt + \sigma_i dW_i(t)$

Note: $k_i = 0.5$ while other parameters are learned from the training data.

the CEV model has the best simulations for the actual data, followed by the MRD; the CIR and HWV models are very close.

Table VI presents the results of arbitrage tests for the non-GBM dynamics, in which most of experimental groups exhibit arbitrage. The CEV model gives the best no-arbitrage performance, showing that 78.65% of CEV-based keywords can be properly priced by using the GBM-based option pricing model. About 53.05% of the CIR model and about 43% of the MRD or HWV model–based options have no arbitrage. For 1-keyword options, the fairness percentage is more than 85% across all experimental groups. However, this number drops to around 38% for multi-keyword options (36.27% for 2-keyword options and 42% for 3-keyword options). For the identified arbitrage, many groups (especially 1-keyword options) show small arbitrage values (around 10%), while arbitrage explodes in some groups.

In summary, Tables IV and VI illustrate that our option pricing methods are effective and reasonably robust for the real sponsored search data. As shown in Figure 7, when the keywords' prices follow a GBM (15.73%), the pricing model ensures that 95.17% of ad options are fairly priced under the 5% arbitrage precision. For the non-GBM keywords, the CEV model is the best performance model, giving 78.65% of fairness; the CIR model is the worst performance model, with only 31.97% of fairness. Overall, the best expected fairness for all keywords is 81.25%, while the worst is 41.91%. We find that the increase of the number of candidate keywords in an ad option increases the likelihood of arbitrage. This is confirmed by the fact that expected fairness drops from 86.83% (99.76% GBM and 83.60% non-GBM for 1-keyword options) to 43.69% (2-keyword options) and 53.39% (3-keyword options), respectively.

4.4. Effects on Search Engine's Revenue

Let us start with the case of 1-keyword options. The example of keyword 'canon cameras' in Figure 8 (a) illustrates (other keywords exhibit the similar pattern) the conclusions

Table VI. Overview of Delta Hedging Arbitrage Testing for non-GBM Dynamics

| Market	n	Group	N	Real data + CEV simu		Real data + MRD simu		Real data + CIR simu		Real data + HWV simu	
				ℙ(arb)	Mean arb	ℙ(arb)	Mean arb	ℙ(arb)	Mean arb	ℙ(arb)	Mean arb
US	1	1	74	2.70%	−1.97%	8.11%	−0.38%	75.68%	1.94%	56.76%	−0.01%
		2	55	0.00%	0.00%	1.82%	1.80%	16.36%	1.86%	9.09%	−0.93%
		3	77	9.09%	−10.26%	5.19%	−7.93%	42.86%	0.85%	23.38%	−1.51%
		4	108	3.70%	1.38%	2.78%	4.27%	7.41%	3.99%	8.33%	2.84%
	2	1	37	24.32%	3.63%	81.08%	−4.09%	97.30%	−16.24%	97.30%	−13.24%
		2	27	37.04%	6.01%	70.37%	5.36%	85.19%	11.01%	85.19%	10.51%
		3	38	31.58%	5.97%	31.58%	−0.41%	73.68%	−6.91%	57.89%	−5.96%
		4	54	29.63%	5.95%	81.48%	6.61%	94.44%	16.78%	94.44%	16.25%
	3	1	24	45.83%	−1.36%	79.17%	−6.04%	100.00%	−19.99%	100.00%	−17.44%
		2	18	11.11%	−2.00%	22.22%	−5.11%	55.56%	−4.39%	72.22%	−2.22%
		3	25	24.00%	7.01%	32.00%	−3.71%	84.00%	−11.14%	76.00%	−10.26%
		4	36	16.67%	3.91%	30.56%	2.66%	83.33%	2.73%	88.89%	3.38%
UK	1	1	58	0.00%	0.00%	0.00%	0.00%	74.14%	1.95%	55.17%	−1.60%
		2	35	0.00%	0.00%	2.86%	1.51%	22.86%	1.65%	14.29%	2.29%
		3	72	5.56%	−5.78%	1.39%	−10.38%	29.17%	1.03%	18.06%	−0.32%
		4	50	4.00%	5.55%	6.00%	4.47%	10.00%	4.56%	8.00%	3.45%
	2	1	29	37.93%	−0.76%	62.07%	−5.52%	89.66%	−14.55%	72.41%	−11.39%
		2	17	47.06%	2.18%	82.35%	5.00%	100.00%	9.87%	100.00%	8.62%
		3	36	19.44%	2.71%	33.33%	−1.78%	75.00%	−5.24%	61.11%	−3.58%
		4	25	64.00%	6.71%	96.00%	9.01%	92.00%	21.17%	92.00%	20.04%
	3	1	19	26.32%	−1.56%	84.21%	−5.21%	100.00%	−16.33%	78.95%	−16.34%
		2	11	18.18%	0.40%	18.18%	−1.09%	63.64%	−1.28%	63.64%	−1.05%
		3	24	16.67%	3.45%	25.00%	−1.61%	79.17%	−9.14%	66.67%	−9.23%
		4	16	37.50%	7.83%	43.75%	7.37%	81.25%	0.68%	81.25%	8.64%

Note: Same notations as in Table IV.
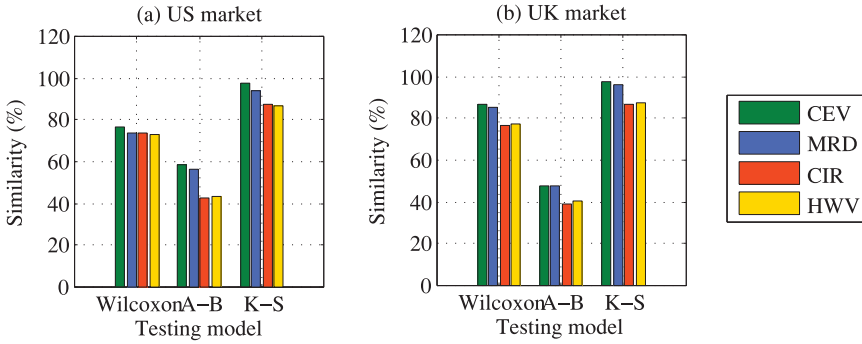
Fig. 6. Overview of model similarity tests: Wilcoxon test, Ansari-Bradley (A-B) test and Two-sample Kolmogorov-Smirnov (K-S) test.
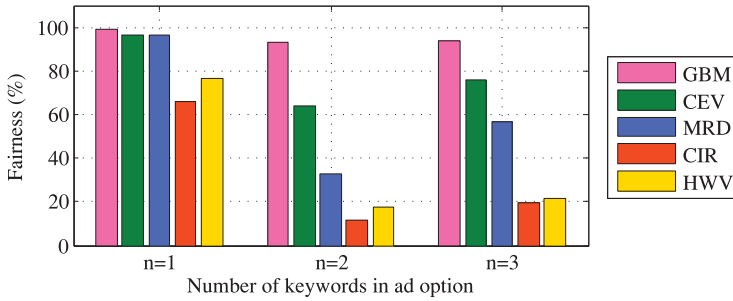

Fig. 7. Overview of pricing model robust tests.

from our theoretical analysis in Section 3.3, that (i) the revenue difference between option and auction is always positive and (ii) that when the fixed CPC $F = \mathbb{E}_0^{\mathbb{Q}}[C(T)]$, the revenue difference $D(F)$ achieves its maximum and the two boundary values are approximately zero.

The non-GBM cases are further examined in Figures 8(b) through 8(e), which show that when the fixed CPC is close to zero, the revenue difference $D(F) \to 0$. This is because when the fixed CPC approximates zero, it is almost certain that the option will be used in the contract period. As such, the only income for the keyword is from the option price, which, in this case, is close to the CPC in the auction market. On the other hand, if the fixed CPC is very high, it is almost certain that the option will not be used. In this case, the option price $\pi_0 \to 0$ and the probability of exercising the option $\mathbb{P}(\mathbb{E}_0^{\mathbb{Q}}[C(T)] \geq F) \to 0$. Hence, $D(F)$ is zero. However, under the non-GBM dynamics, the point $F = \mathbb{E}_0^{\mathbb{Q}}[C(T)]$ is not the optimal value that gives the maximum $D(F)$, which indicates that arbitrage may occur.

Next, Figure 9 illustrates an empirical example of a 2-keyword ad option. The candidate keywords are 'non profit debt consolidation' and 'canon 5d'. Figure 9(a) shows that the higher the fixed CPCs, the lower is the option price (even though the option price is less sensitive to the keyword 'canon 5d') and it achieves the maximum when all the fixed CPCs are zeros. These monotone results are the same as the 1-keyword options. Figure 9(b) then shows the revenue difference curve of the search engine, in which the red star represents the value where $F_1 = \mathbb{E}_0^{\mathbb{Q}}[C_1(T)]$ and $F_2 = \mathbb{E}_0^{\mathbb{Q}}[C_2(T)]$. The expected revenue differences are all nonnegative, showing that this 2-keyword ad
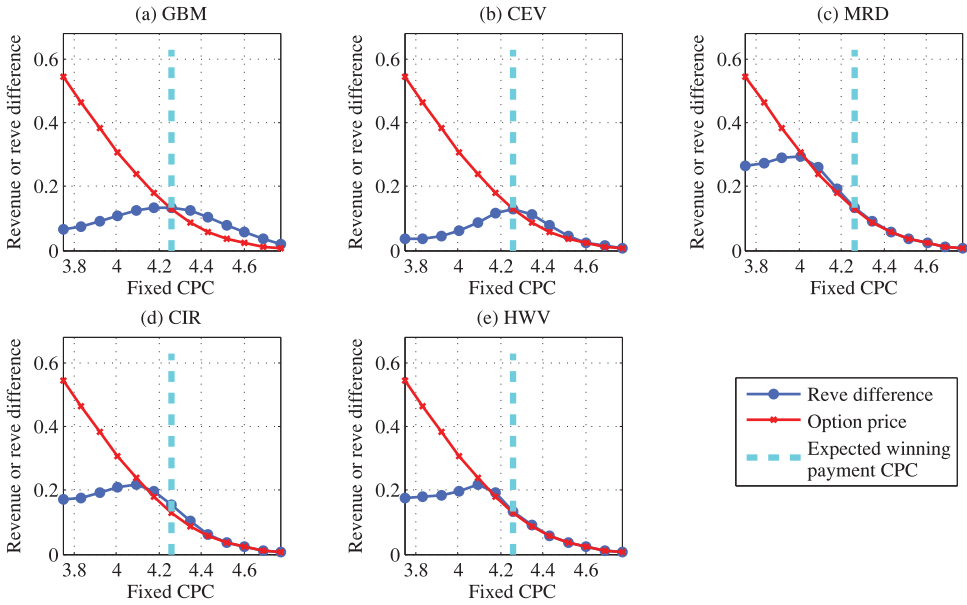
Fig. 8. Empirical example of analysing the search engine's revenue for the keyword 'canon cameras'.
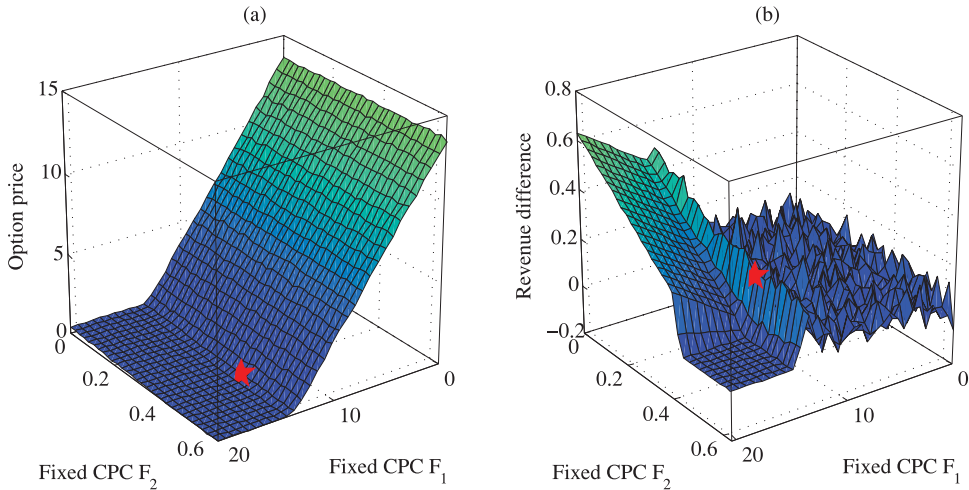

Fig. 9. Empirical example of analysing the search engine's revenue for the keywords 'non profit debt consolidation' and 'canon 5d', where $\rho = 0.2247$.

option is beneficial to the search engine's revenue. However, the red star point is not the maximum difference revenue. This is different from the 1-keyword ad options.

For higher-dimensional ad options (i.e., $n \geq 3$), it is not possible to graphically examine the revenue difference. However, based on the earlier discussions, two findings can be summarised. First, there are boundary values of the revenue differences: if every $F_i \to 0$, $D(\mathbf{F}) \to 0$; and if every $F_i \to \infty$, $D(\mathbf{F}) \to 0$. Second, there exists a maximum revenue difference value even though this may not at the point where $F_i = \mathbb{E}_0^{\mathbb{Q}}[C_i(T)]$. Hence, compared to only keyword auctions, properly setting the fixed CPCs can increase the search engine's expected revenue.

5. CONCLUDING REMARKS

In this article, we proposed a novel framework to provide flexible guaranteed deliveries for sponsored search, from which both buy and sell sides can benefit. On the buy side, advertisers are able to secure a certain number of clicks from their targeted keywords in the future and can decide how to advertise later. They can be released from auction campaigns and can manage price risk under the given budgets. On the sell side, the search engine can sell the future clicks in advance and can receive a more stable and increased expected revenue over time. In addition, advertisers would be more loyal to a search engine due to the contractual relationships, which has the potential to boost the search engine's revenue on the long run.

We also believe that the proposed ad options will soon be welcomed by the sponsored search market. Several similar but different developments that have appeared in the display digital markets support our point of view. They are:

> *09/2013:* AOL's Programmatic Upfront[4]
> *03/2013:* OpenX Programmatic Guarantee [OpenX 2013]
> *10/2012:* Adslot Media's Programmatic Direct Media Buying[5]
> *10/2012:* Shiny Ads Programmatic Direct Advertising Platform[6]
> *10/2012:* iSOCKET's Programmatic Direct[7]

Our work differs from these developments in many aspects. First, we focus on sponsored search, while they are for display advertising. Second, the proposed ad options provide flexible guaranteed deliveries (e.g., multi-keyword targeting, multi-click exercise, early exercise, no obligation of exercise) while other recent developments do not provide such features.

Our work leaves several directions for future research. First, to address the limitations of GBM, other stochastic processes tailored to some specific keywords are worth studying, such as the jump-diffusion model [Kou 2002] and the stochastic volatility model [Chen and Wang 2015]. The most challenging part of this future research is that the underlying model is multi-dimensional and needs to be computational fast. Second, it would be interesting to discuss an optimal pricing and allocation model of ad options so that a search engine can algorithmically manipulate the limited future clicks in front of uncertain demand. Third, the game-theoretical pricing of ad options can be another direction.

APPENDICES

A. PROOF OF THE NO-EARLY EXERCISE PROPERTY FOR THE PROPOSED AD OPTION

Equation (3) can be rewritten as $\Phi(\boldsymbol{x}) = \max\{\boldsymbol{x} - \boldsymbol{f}, 0\}$, where $\boldsymbol{x}' = [x_1, \ldots, x_n]$ and $\boldsymbol{f}' = [f_1, \ldots, f_n]$. It is not difficult to find that $\Phi(\boldsymbol{x})$ is multivariate convex. Let $0 \leq \lambda \leq 1$ and let $\boldsymbol{y}' = [y_1, \ldots, y_n]$, if the elements of vector $\boldsymbol{a} = \boldsymbol{y} - \boldsymbol{x}$ are all nonnegative, then

$$\Phi(\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \leq \lambda\Phi(\boldsymbol{x}) + (1 - \lambda)\Phi(\boldsymbol{y}).$$

If $\boldsymbol{y}' = (0, \ldots, 0)$, and using the fact that $\Phi(\boldsymbol{0}) = 0$, we obtain

$$\Phi(\lambda\boldsymbol{x}) \leq \lambda\Phi(\boldsymbol{x}), \quad \text{for all } x_i \geq 0, \ 0 \leq \lambda \leq 1.$$

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[4]http://www.aolplatforms.com/aolupfront2014.

[5]http://www.automatedguaranteed.com.

[6]https://shinyads.com/solutions/direct-programmatic-guaranteed.

[7]https://www.isocket.com/automated-guaranteed.

For $0 \le s \le t \le T$, since $0 \le e^{-r(t-s)} \le 1$, we then have

$$
\begin{aligned}
\mathbb{E}_s^{\mathbb{Q}}\big[e^{-r(t-s)}\Phi(\boldsymbol{X}(t))\big] &\ge \mathbb{E}_s^{\mathbb{Q}}\big[\Phi\big(e^{-r(t-s)}\boldsymbol{X}(t)\big)\big] \\
&\ge \Phi\big(\mathbb{E}_s^{\mathbb{Q}}\big[e^{-r(t-s)}\boldsymbol{X}(t)\big]\big) \;\; \text{(by the Jensen's Inequality)} \\
&= \Phi\big(e^{rs}\mathbb{E}_s^{\mathbb{Q}}\big[e^{-rt}\boldsymbol{X}(t)\big]\big),
\end{aligned}
$$

where $\mathbb{E}_s^{\mathbb{Q}}[\cdot]$ is the conditional expectation with respect to time s under the risk-neutral probability measure $\mathbb{Q}$. Since $e^{-rt}\boldsymbol{X}(t)$ is a martingale under $\mathbb{Q}$ [Björk 2009], then

$$
\Phi\big(e^{rs}\mathbb{E}_s^{\mathbb{Q}}\big[e^{-rt}\boldsymbol{X}(t)\big]\big) = \Phi\big(e^{rs}e^{-rs}\boldsymbol{X}(s)\big) = \Phi\big(\boldsymbol{X}(s)\big).
$$

Hence, $\mathbb{E}_s^{\mathbb{Q}}[e^{-r(t-s)}\Phi(\boldsymbol{X}(t))] \ge \Phi(\boldsymbol{X}(s))$, showing that $e^{-rt}\Phi(\boldsymbol{X}(t))$ is a submartingale under $\mathbb{Q}$. This indicates that the proposed ad option can be priced the same as its European structure, focusing on the payoff on the contract expiration date. For further detailed discussions about martingale and submartingale, see Björk [2009].

B. DERIVATION OF THE AD OPTION PRICING FORMULA

Since the proposed ad option complements the existing keyword auctions, there may exist a situation that some advertisers want to make guaranteed profits only from the difference of costs between option and auction markets without taking any risk. This situation is called *arbitrage* [Varian 1987; Björk 2009]. Hence, we must fairly evaluate the option so that arbitrage is eliminated.

In the context of sponsored search, we consider that an advertiser buys an n-keyword m-click ad option at time 0. Then, at time t, $t \in [0, T]$, the difference between the option value and the market value of candidate keywords can be expressed as

$$
\Pi(t) = V(t, \boldsymbol{C}(t); \boldsymbol{F}, T, m) - \sum_{i=1}^n \psi_i(t)C_i(t), \tag{19}
$$

where $\psi_i(t)$ represents the number of clicks needed for the keyword K_i such that $\sum_i \psi_i(t) = m$. Here, we call $\Pi(t)$ the *value difference process*. Recall that, in Equation (3), we consider the value of an n-keyword m-click option as the sum of m independent n-keyword 1-click options; for mathematical convenience, Equation (19) can be rewritten as follows:

$$
\Pi(t) = m\left(V(t, \boldsymbol{C}(t); \boldsymbol{F}, T, 1) - \sum_{i=1}^n \Delta_i C_i(t)\right), \tag{20}
$$

where Δ_i represents the probability that a single click goes for the keyword K_i and $\sum_{i=1}^n \Delta_i = 1$. The changes of Π over a sufficient small period of time dt is then

$$
d\Pi(t) = m\left(\frac{\partial V}{\partial t}dt + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \sigma_i\sigma_j\rho_{ij}C_iC_j\frac{\partial^2 V}{\partial C_i\partial C_j}dt + \sum_{i=1}^n \frac{\partial V}{\partial C_i}dC_i - \sum_{i=1}^n \Delta_i dC_i\right). \tag{21}
$$

The uncertain components in $d\Pi(t)$ can be removed if $\Delta_i = \partial V/\partial C_i$. This is called *delta hedging* in option pricing theory [Wilmott 2006]. Hence, $\Pi(t)$ now becomes a riskless process over time:

$$
d\Pi(t) = m\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \sigma_i\sigma_j\rho_{ij}C_iC_j\frac{\partial^2 V}{\partial C_i\partial C_j}\right)dt. \tag{22}
$$

We assume that the advertiser has no initial fund and borrows the money from others at the riskless bank interest rate r; thus the interest of this borrowing is

$$d\Pi(t) = r\Pi(t)dt = rm\left(V - \sum_{i=1}^{n} \frac{\partial V}{\partial C_i}C_i\right)dt. \tag{23}$$

Equations (22) and (23) need to be equal; otherwise, arbitrage exists. If the riskless growth rate of the value difference process is larger than the riskless bank interest rate, the advertiser can obtain arbitrage by: (i) borrowing the money from the bank at interest rate r to buy an ad option first or (ii) selling the ad option later to repay the bank interest. In the case in which the riskless growth rate of the value difference process is smaller than the riskless bank interest rate, the advertiser can obtain the riskless surplus by: (i) selling short an ad option first and saving the revenue in a bank account or (ii) using the deposit money to buy the clicks of underlying keywords later. In either case, the advertiser can finally receive a riskless surplus; therefore, arbitrage exists.

Solving Equations (22) and (23) can give a parabolic partial differential equation (PDE) for the no-arbitrage equilibrium:

$$\frac{\partial V}{\partial t} + r\sum_{i=1}^{n} \frac{\partial V}{\partial C_i}C_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 V}{\partial C_i \partial C_j}\sigma_i\sigma_j\rho_{ij}C_iC_j - rV = 0.$$

This PDE satisfies the boundary condition in Equation (3). By employing the multidimensional Feynman-Kač stochastic representation [Björk 2009], we obtain the solution

$$V(t, \boldsymbol{C}(t); \boldsymbol{F}, T, 1) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[\Phi(\boldsymbol{C}(T))],$$

where $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ is the conditional expectation with respect to time t under the risk-neutral probability $\mathbb{Q}$. The process $C_i(t)$ can be rewritten as

$$dC_i(t) = rC_i(t)dt + \sigma_i C_i(t)dW_i^{\mathbb{Q}}(t),$$

where $W_i^{\mathbb{Q}}(t)$ is the standard Brownian motion under $\mathbb{Q}$. Therefore, the option price π_0 can be calculated by the following formula:

$$\pi_0 = V(0, \boldsymbol{C}(0); \boldsymbol{F}, T, m) = mV(0, \boldsymbol{C}(0); \boldsymbol{F}, T, 1) = me^{-rT}\mathbb{E}_0^{\mathbb{Q}}[\Phi(\boldsymbol{C}(T))].$$

C. OPTION PRICING FORMULAS FOR SPECIAL CASES

If $n = 1$, Equation (6) is equivalent to the Black-Scholes-Merton (BSM) pricing formula for a European call option [Black and Scholes 1973; Merton 1973]. Then we have

$$\pi_0 = mC(0)\mathcal{N}[\zeta_1] - mFe^{-rT}\mathcal{N}[\zeta_2], \tag{24}$$

where $\zeta_1 = \frac{1}{\sigma\sqrt{T}}(\ln\{C(0)/F\} + (r + \frac{\sigma^2}{2})T)$ and $\zeta_2 = \zeta_1 - \sigma\sqrt{T}$.

If $n = 2$, Equation (6) contains a bivariate normal distribution. Hence, we can calculate the option price the same way as the dual-strike European call option

[Zhang 1998]:

$$\pi_0 = mC_1(0) \int_{-\infty}^{\zeta_1 + \sigma_1 \sqrt{T}} f(u) \mathcal{N} \left[\frac{q_1(u + \sigma_1 \sqrt{T}) - \rho \sigma_1 \sqrt{T} + \rho u}{\sqrt{1 - \rho^2}} \right] du$$

$$+ mC_2(0) \int_{-\infty}^{\zeta_2 + \sigma_2 \sqrt{T}} f(v) \mathcal{N} \left[\frac{q_2(u + \sigma_2 \sqrt{T}) - \rho \sigma_1 \sqrt{T} + \rho v}{\sqrt{1 - \rho^2}} \right] dv$$

$$- me^{-rT} \left(F_1 \int_{-\infty}^{\zeta_1} f(u) \mathcal{N} \left[\frac{q_1(u) + \rho u}{\sqrt{1 - \rho^2}} \right] du + F_2 \int_{-\infty}^{\zeta_2} f(v) \mathcal{N} \left[\frac{q_2(v) + \rho v}{\sqrt{1 - \rho^2}} \right] dv \right), \quad (25)$$

where

$$q_1(u) = \frac{1}{\sigma_2 \sqrt{T}} \left(\ln \left\{ \frac{F_2 - F_1 + C_1(0) e^{(r - \frac{1}{2}\sigma_1^2)T - u\sigma_1 \sqrt{T}}}{C_2(0)} \right\} - \left(r - \frac{1}{2}\sigma_2^2 \right) T \right),$$

$$q_2(u) = \frac{1}{\sigma_1 \sqrt{T}} \left(\ln \left\{ \frac{F_1 - F_2 + C_2(0) e^{(r - \frac{1}{2}\sigma_2^2)T - v\sigma_2 \sqrt{T}}}{C_1(0)} \right\} - \left(r - \frac{1}{2}\sigma_1^2 \right) T \right),$$

$$\zeta_1 = \frac{1}{\sigma_1 \sqrt{T}} \left(\ln\{C_1(0)/F_1\} + \left(r - \frac{1}{2}\sigma_1^2 \right) T \right),$$

$$\zeta_2 = \frac{1}{\sigma_2 \sqrt{T}} \left(\ln\{C_2(0)/F_2\} + \left(r - \frac{1}{2}\sigma_2^2 \right) T \right).$$

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