Pricing average price advertising options when underlying spot market prices are discontinuous[1]

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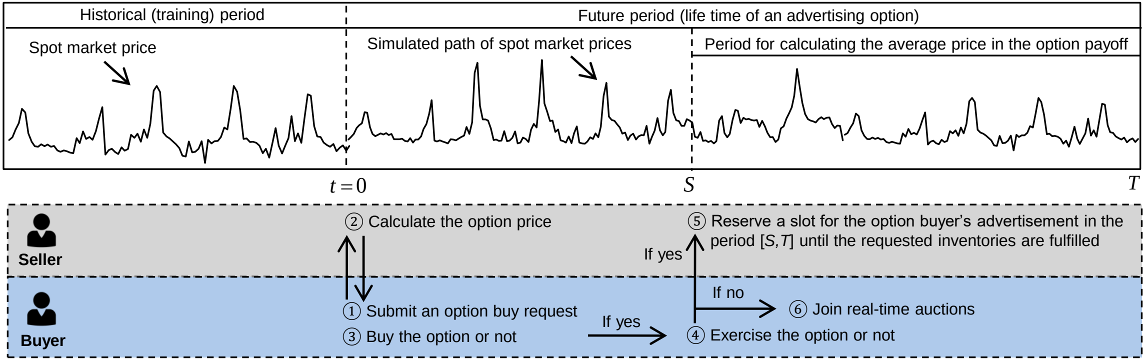
Average price advertising option



Jump-diffusion stochastic process

Given $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$, the spot market price $X(t)$ follows the SDE

$$\frac{dX(t)}{X(t^-)} = \underbrace{\mu dt + \sigma dW(t)}_{\substack{\text{Continuous}\\ \text{component}}} + \underbrace{d\left(\sum_{i=1}^{N(t)}(Y_i - 1)\right)}_{\substack{\text{Discontinuous}\\ \text{component}}}, \tag{1}$$

- μ and σ are the constant drift and volatility terms
- $X(t^-)$ stands for the value of X just before a jump at time t
- $N(t)$ is a homogeneous Poisson process with intensity λ
- $\{Y_i, i = 1, \cdots\}$ is a sequence of i.i.d. non-negative variables representing the jump sizes
- $N(t)$, $W(t)$, Y_i are assumed to be independent

Choices of jump size distributions

Let $V_i = \ln\{Y_i\}$, then

- $V_i \sim \mathbf{N}(\alpha, \beta^2)$, then $\mathbb{E}[e^{V_i}] = e^{\alpha + \frac{1}{2}\beta^2}$.
- $V_i \sim \mathbf{ADE}(\eta_1, \eta_2, p_1, p_2)$, then $\mathbb{E}[e^{V_i}] = p_1 \frac{\eta_1}{\eta_1 - 1} + p_2 \frac{\eta_2}{\eta_2 + 1}$.
- $V_i \sim \mathbf{LAP}(\varrho, \eta)$, then $\mathbb{E}[e^{V_i}] = \frac{e^{\varrho}}{1 - \eta^2}$.

Solution to Eq.(1)

By checking Itô calculus, we have

$$X(t) = X(0) \exp\left\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\right\} \prod_{i=1}^{N(t)} Y_i, \tag{2}$$

where $\prod_{i=1}^{0} = 1$. Hence, it is an exponential Lévy model.

Option payoff based on power mean and CTR

$$\Phi(\mathbf{X}) = \theta \left(\frac{\widetilde{c}}{c} \underbrace{\left(\frac{1}{m} \sum_{i=\widetilde{m}+1}^{\widetilde{m}+m} X_i^{\gamma} \right)^{\frac{1}{\gamma}}}_{:= \psi(\gamma|\mathbf{X})} - K \right)^{+}, \tag{3}$$

- $(\cdot)^{+} := \max\{\cdot, 0\}$
- θ is the requested number of impressions or clicks
- c is the CTR of the option buyer's advertisement
- $\widetilde{c}$ is the average CTR of relevant or similar advertisements
- K is the exercise price which can be a fixed CPM or CPC
- $\mathbf{X}$ is a vector of the spot market prices in the future period $[S, T]$
- $\left(\frac{1}{m} \sum_{i=\widetilde{m}+1}^{\widetilde{m}+m} X_i^{\gamma}\right)^{1/\gamma}$ is the power mean of these prices.

Special and limiting cases of power mean

γ	$\psi(\gamma\|\mathbf{X})$	Description
$-\infty$	$\min\left\{X_{\widetilde{m}+1},\cdots,X_{\widetilde{m}+m}\right\}$	Minimum value
-1	$m/\left(\frac{1}{X_{\widetilde{m}+1}}+\cdots+\frac{1}{X_{\widetilde{m}+m}}\right)$	Harmonic mean
0	$\left(\prod_{i=\widetilde{m}+1}^{\widetilde{m}+m} X_i\right)^{\frac{1}{m}}$	Geometric mean
1	$\frac{1}{m}\sum_{i=\widetilde{m}+1}^{\widetilde{m}+m} X_i$	Arithmetic mean
2	$\left(\frac{1}{m}\sum_{i=\widetilde{m}+1}^{\widetilde{m}+m} X_i^2\right)^{\frac{1}{2}}$	Quadratic mean
∞	$\max\left\{X_{\widetilde{m}+1},\cdots,X_{\widetilde{m}+m}\right\}$	Maximum value

Monotonicity property: if $\gamma_1 \leq \gamma_2$, then $\psi(\gamma_1|\mathbf{X}) \leq \psi(\gamma_2|\mathbf{X})$

Solution to Eq.(1) under $\mathbb{Q}$

The solution to Eq. (1) under the risk-neutral probability measure $\mathbb{Q}$ is

$$X(t) = X(0) \exp\left\{ (r - \lambda\zeta - \frac{1}{2}\sigma^2)t + \sigma W(t) \right\} \prod_{i=1}^{N(t)} Y_i, \tag{4}$$

where $\zeta := \mathbb{E}[e^{V_i}] - 1$, and its detailed calculation is given in the following table.

Distribution of Y_i	Distribution of V_i	$\zeta := \mathbb{E}[e^{V_i}] - 1$
Log-normal	$\mathbf{N}(\alpha, \beta^2)$	$e^{\alpha + \frac{1}{2}\beta^2} - 1$
Log-ADE	$\mathbf{ADE}(\eta_1, \eta_2, p_1, p_2)$	$p_1 \frac{\eta_1}{\eta_1 - 1} + p_2 \frac{\eta_2}{\eta_2 + 1} - 1$
Log-laplacian	$\mathbf{LAP}(\varrho, \eta)$	$\frac{e^{\varrho}}{1 - \eta^2} - 1$

Option price

The option price can be obtained as follows:

$$\pi_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\Phi(\mathbf{X})|\mathcal{F}_0], \tag{5}$$

where $\mathbb{E}^{\mathbb{Q}}[\cdot|\mathcal{F}_0]$ represents the expectation conditioned on the information up to time 0 under the risk-neutral probability measure $\mathbb{Q}$.

General solution[3]

Input: $X(0), r, \sigma, S, T, m, K, c, \widetilde{c}, z, \theta, \gamma,$ ˘
(where ˘ $= \{\alpha, \beta\}$ or $\{\eta_1, \eta_2, p_1, p_2\}$ or $\{\varrho, \eta\}$)

$\Delta t \leftarrow \frac{1}{m}(T - S)$; $\widetilde{m} \leftarrow \lceil \frac{S}{\Delta t} \rceil$; $\zeta \leftarrow$ Table 2;
for $j \leftarrow 1$ to z **do**
    $X_0^{\{j\}} \leftarrow X(0)$;
    for $i \leftarrow 1$ to $\widetilde{m} + m$ **do**
        $a_i \leftarrow \mathbf{N}\big((r - \lambda\zeta - \frac{1}{2}\sigma^2)\Delta t, \sigma^2 \Delta t\big)$;
        $\xi_i \leftarrow \mathbf{BER}(\lambda \Delta t)$;
        $v_i \leftarrow \mathbf{N}(\alpha, \beta^2)$ or $\mathbf{ADE}(\eta_1, \eta_2, p_1, p_2)$ or $\mathbf{LAP}(\varrho, \eta)$;
        $\ln\{X_i^{\{j\}}\} \leftarrow \ln\{X_{i-1}^{\{j\}}\} + a_i + \xi_i v_i$;
    end for
    $\Phi^{\{j\}} \leftarrow$ Eq. (3);
end for
$\pi_0 \leftarrow e^{-rT}\big(\frac{1}{z}\sum_{j=1}^{z} \Phi^{\{j\}}\big)$.

Output: π_0

[3] See Algorithm 1.

Special case[4]

If $\gamma = 0$ and $V_i \sim \mathbf{N}(\alpha, \beta^2)$, the option price π_0 can be obtained by the formula

$$\pi_0 = \theta e^{-(r+\lambda)T} \sum_{k=0}^{\infty} \frac{(\lambda T)^k}{k!} \left(\frac{\widetilde{c}}{c} X(0) \Omega \mathscr{N}(\xi_1) - K \mathscr{N}(\xi_2) \right), \tag{6}$$

where $\mathscr{N}(\cdot)$ is the cumulative standard normal distribution function, and

$$\begin{aligned} A &= \frac{1}{2}(r - \lambda\zeta - \frac{1}{2}\sigma^2)(T + S) + k\alpha, \\ B^2 &= \frac{1}{3}\sigma^2 T + \frac{2}{3}\sigma^2 S + k\beta^2, \\ \Omega &= e^{\frac{1}{2}(B^2 + 2A)}, \qquad \phi = \ln\{cK\} - \ln\{\widetilde{c} X(0)\}, \\ \xi_1 &= B - \frac{\phi}{B} + \frac{A}{B}, \qquad \xi_2 = \frac{A}{B} - \frac{\phi}{B}. \end{aligned}$$

[4] See Theorem 1 and its proof.

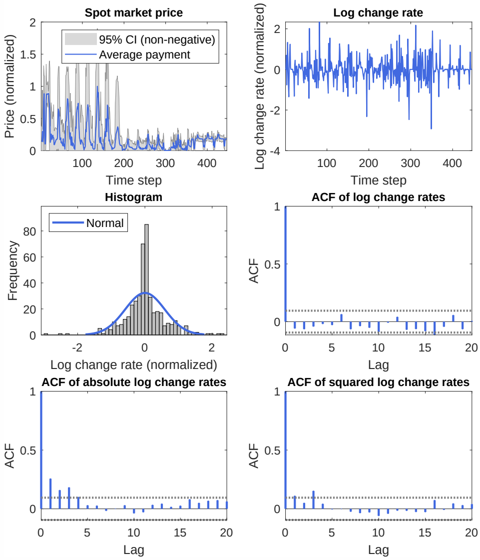
Data

Dataset	SSP	Google UK	Google US
Advertising type	Display	Search	Search
Auction model	SP	GSP	GSP
Advertising position	NA	1st position†	1st position†
Bid quote	GBP/CPM	GBP/CPC	GBP/CPC
Market of targeted users‡	UK	UK	US
Time start	08/01/2013	26/11/2011	26/11/2011
Time end	14/02/2013	14/01/2013	14/01/2013
# of total advertising slots	31	106	141
Data reported frequency	Auction	Day	Day
# of total auctions	6,646,643	NA	NA
# of total bids	33,043,127	NA	NA

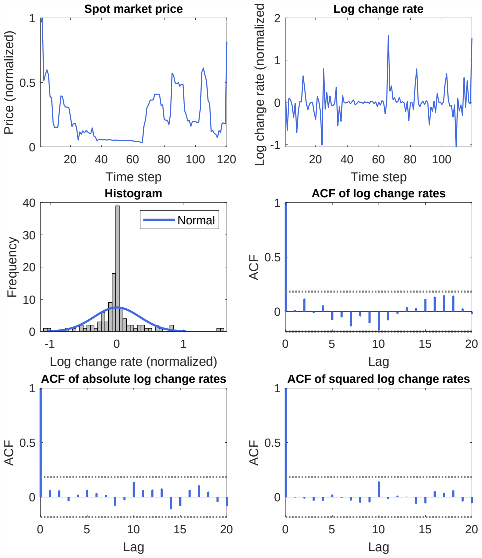
†In the mainline paid listing of the SERP. ‡Market by geographical areas.

Stylized facts

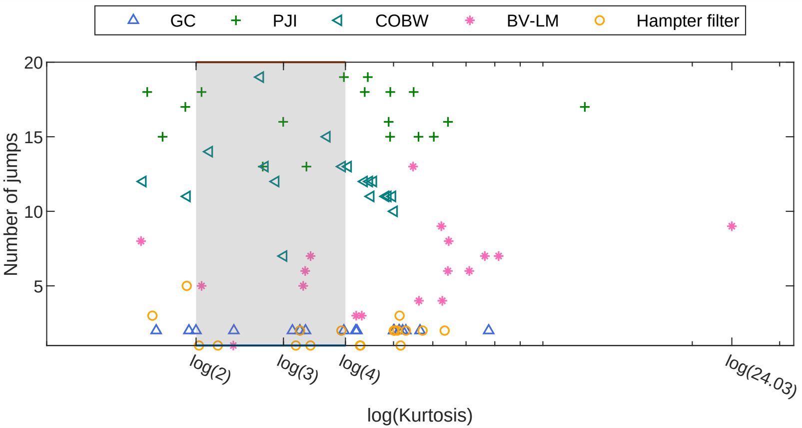
An ad slot (hourly) in the SSP dataset



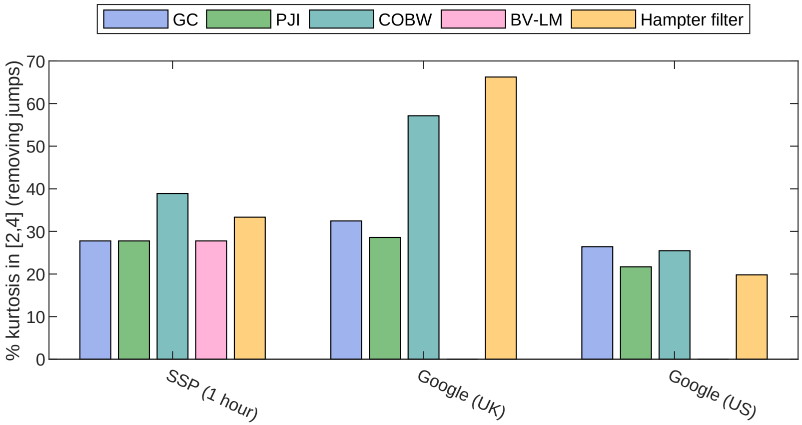
Keyword “notebook laptops” (daily) in Google US dataset



Price jump detection methods on the SSP dataset (hourly)

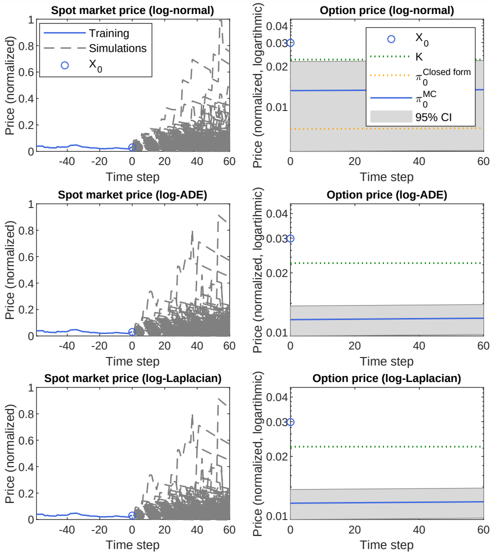


Price jump detection methods on the SSP and Google datasets



Note: the BV-LM is not used for Google datasets as there are no intra-day campaign records.

Option pricing for keyword "panasonic dmc"



Thank you!

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Appendix I: proof of Theorem 1

The geometric mean $\psi(\gamma = 0|\mathbf{X})$ can be rewritten in a continuous-time form

$$\psi(\gamma = 0|\mathbf{X}) = \exp\left\{\frac{1}{T-S}\int_S^T \ln\{X(t)\}dt\right\},$$

then

$$Z(T)|N(T) = k \sim \mathbf{N}\Big((r - \lambda\zeta - \frac{1}{2}\sigma^2)T + k\alpha, \sigma^2 T + k\beta^2\Big).$$

Below we show $\psi(0|\mathbf{X})$ is log-normally distributed.

$$\begin{aligned}
&\psi(0|\mathbf{X}) \\
&= X_0\left(\prod_{i=\widetilde{m}+1}^{\widetilde{m}+m} X_i/X_0^m\right)^{1/m} \\
&= X_0 \exp\left\{\frac{1}{m}\ln\left\{\left(\frac{X_{\widetilde{m}}}{X_0}\right)^m\left(\frac{X_{\widetilde{m}+1}}{X_{\widetilde{m}}}\right)^{m}\cdots\left(\frac{X_{\widetilde{m}+m}}{X_{\widetilde{m}+m-1}}\right)\right\}\right\}
\end{aligned}$$

Appendix I: proof of Theorem 1

Since $\Delta t = \frac{T-S}{m}$, so $\widetilde{m} = \frac{S}{\Delta t} = \frac{S}{T-S} m$, and then

$$\ln \left\{ \frac{X_{\widetilde{m}}}{X_0} \right\} \Bigg|_{N(T)=k} \sim \mathbf{N}\Big((r - \lambda\zeta - \frac{1}{2}\sigma^2) S + k\alpha, \sigma^2 S + k\beta^2 \Big),$$

and for $i = 0, \cdots, (m-1)$,

$$\ln \left\{ \frac{X_{\widetilde{m}+i+1}}{X_{\widetilde{m}+i}} \right\} \Bigg|_{N(T)=k} \sim \mathbf{N}\Big((r - \lambda\zeta - \frac{1}{2}\sigma^2) \Delta t, \sigma^2 \Delta t \Big).$$

Let $\Theta = \frac{1}{T-S} \int_S^T Z(t) dt$, then $\Theta\big|_{N(T)=k} \sim \mathbf{N}\big(\widetilde{A}, \widetilde{B}^2\big)$, where

$$\widetilde{A} = (r - \lambda\zeta - \frac{1}{2}\sigma^2)(\frac{(m+1)}{m} \frac{T-S}{2} + S) + k\alpha,$$

$$\widetilde{B}^2 = \frac{(m+1)(2m+1)}{6m^2} \sigma^2 (T-S) + \sigma^2 S + k\beta^2.$$

Appendix I: proof of Theorem 1

If $m \to \infty$, $\Theta\big|_{N(T)=k} \sim \mathbf{N}\big(A, B^2\big)$, where

$$A = \frac{1}{2}(r - \lambda\zeta - \frac{1}{2}\sigma^2)(T + S) + k\alpha,$$
$$B^2 = \frac{1}{3}\sigma^2 T + \frac{2}{3}\sigma^2 S + k\beta^2.$$

Hence, the option price can be obtained as

$$\begin{aligned}
\pi_0 &= \theta e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{\widetilde{c}}{c}X_0 e^{\Theta} - K\right)^+ \Big| \, N(T) = k\right]\Big|\mathcal{F}_0\right] \\
&= \theta e^{-rT}\sum_{k=0}^{\infty}\frac{(\lambda T)^k}{k!}e^{-\lambda T}\mathbb{E}_0^{\mathbb{Q}}\left[\left(\frac{\widetilde{c}}{c}X_0 e^{\Theta} - K\right)^+\right] \\
&= \theta e^{-rT}\sum_{k=0}^{\infty}\frac{(\lambda T)^k}{k!}e^{-\lambda T}\int_{\phi}^{\infty}\left(\frac{\widetilde{c}}{c}X_0 e^{\Theta} - K\right)f(\Theta)d\Theta,
\end{aligned}$$

solving the integral terms then completes the proof. $\square$

Appendix II: model parameters estimation

The discretization of Eq. (2) is

$$\frac{X(t)}{X(t-\Delta t)} = \exp\left\{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_t\right\}\prod_{i=1}^{n_t} Y_i, \tag{7}$$

where $\varepsilon_t \sim \mathbf{N}(0,1)$, $n_t = N(t) - N(t-\Delta t)$ is the number of price jumps between $t-\Delta t$ and t.

Appendix II: model parameters estimation

Let $\widetilde{Z}(t) = \ln\{X(t)/X(t-\Delta t)\}$, $\mu_V = \mathbb{E}[V_i]$, $\sigma_V^2 = \mathrm{Var}[V_i]$, $\mu^* = \mu - \frac{1}{2}\sigma^2 + \lambda\mu_V$, and $\Delta J_t^* = \sum_{i=1}^{n_t} V_i - \lambda\mu_V\Delta t$, then we have $\mathbb{E}[\Delta J_t^*] = \mathbb{E}[n_t]\mu_V - \lambda\Delta t\mu_V = 0$, $\mathbb{E}[\Delta J_t^*|n_t] = n_t\mu_V - \lambda\Delta t\mu_V$, $\mathrm{Var}[\Delta J_t^*|n_t] = n_t^2\sigma_V^2$, $\mathbb{E}[\widetilde{Z}(t)|n_t] = (\mu - \frac{1}{2}\sigma^2)\Delta t + n_t\mu_V$, $\mathrm{Var}[\widetilde{Z}(t)|n_t] = \sigma^2\Delta t + n_t^2\sigma_V^2$. For simplicity, $\widetilde{Z}(t)|n_t$ is considered to be normally distributed, then we can maximize the log-likelihood function $\mathscr{L}(\mu^*, \sigma, \mu_V, \sigma_V)$ as follows

$$\underset{\mu^*,\sigma\geq 0,\mu_V,\sigma_V\geq 0}{\arg\max} \ln\left\{\mathscr{L}(\mu^*, \sigma, \mu_V, \sigma_V)\right\} = \ln\left\{\prod_{j=1}^{\widetilde{n}}\sum_{k=0}^{\infty}\mathbb{P}(n_t = k)f(\widetilde{z}_j|n_t)\right\}, \tag{8}$$

where $\widetilde{n}$ is the number of observations, the density $f(\widetilde{z}_j)$ is the sum of the conditional probabilities density $f(\widetilde{z}_j|n_t)$ weighted by the probability of the number of jumps $\mathbb{P}(n_t)$.

Appendix II: model parameters estimation

This is an infinite mixture of normal variables, and there is usually one price jump if Δt is small. Therefore, the estimation becomes:

$$\underset{\sigma\geq 0,\mu_V,\sigma_V\geq 0}{\arg\max} \ln\left\{\prod_{j=1}^{\widetilde{n}}\Big((1-\lambda\Delta t)f_1(\widetilde{z}_j)+\lambda\Delta t f_2(\widetilde{z}_j)\Big)\right\}, \tag{9}$$

where $f_1(\widetilde{z}_j)$ is the density of $\mathbf{N}\big((\mu-\frac{1}{2}\sigma^2)\Delta t, \sigma^2\Delta t\big)$, and $f_2(\widetilde{z}_j)$ is the density of $\mathbf{N}\big((\mu-\frac{1}{2}\sigma^2)\Delta t+\mu_V, \sigma^2\Delta t+\sigma_V^2\big)$.